Trigonometry - Angles and Ratios Lesson #5:
Special Triangles and Exact Values

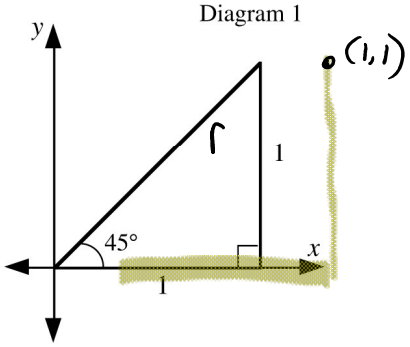
Overview

In this lesson, we will determine the exact value of the sine ratio, the cosine ratio, and the tangent ratio for a given angle with a reference angle of 0°, 30°, 45°, 60°, and 90°.

Investigation

a) Diagram 1 shows an angle of 45° in standard position. An isosceles triangle is drawn whose equal sides are 1 unit.

Diagram 1



i) Determine the length of the hypotenuse.

$r^2 = x^2 + y^2$ $\quad r^2 = 1^2 + 1^2$ $\quad r = \sqrt{2}$

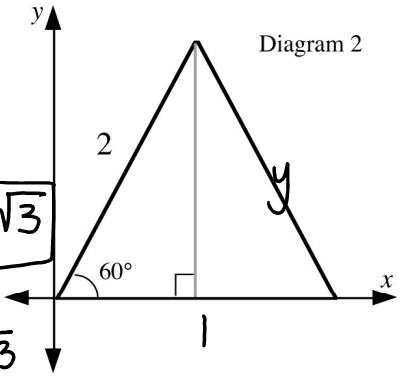
$r^2 = 2$

ii) Use SOHCAHTOA or the x, y, r formulas to complete:

sin 45° = $\frac{y}{r}$ $= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad$ cos 45° = $\frac{x}{r}$ $= \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad$ tan 45° = $\frac{y}{x}$ $= \frac{1}{1} = 1$

b) Diagram 2 shows an angle of 60° in standard position. An equilateral triangle is drawn whose equal sides are 2 units, and a vertical altitude is drawn which divides the equilateral triangle into two congruent triangles.

Diagram 2



i) Determine the length of the altitude.

$y^2 = r^2 - x^2$ $\quad y^2 = 2^2 - 1^2$ $\quad y = \sqrt{3}$

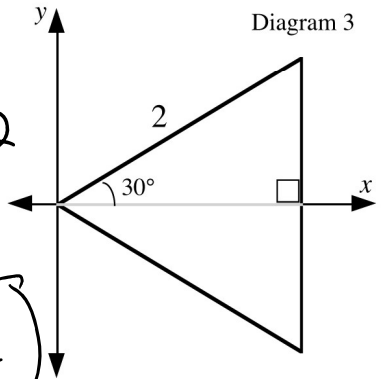
$y^2 = 3$

ii) Complete :

sin 60° = $\frac{y}{r}$ $= \frac{\sqrt{3}}{2}$ $\quad$ cos 60° = $\frac{x}{r}$ $= \frac{1}{2}$ $\quad$ tan 60° = $\frac{y}{x}$ $= \frac{\sqrt{3}}{1} = \sqrt{3}$

c) Diagram 3 shows an angle of 30° in standard position. An equilateral triangle is drawn whose equal sides are 2 units, and a horizontal altitude is drawn which divides the equilateral triangle into two congruent triangles.

Diagram 3



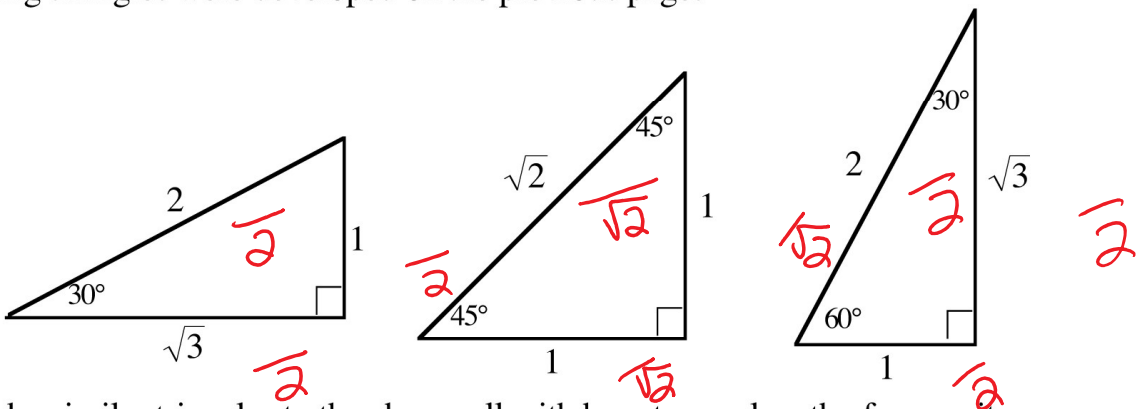
$x = \sqrt{3}$ $\quad y = 1$ $\quad r = 2$

i) Complete:

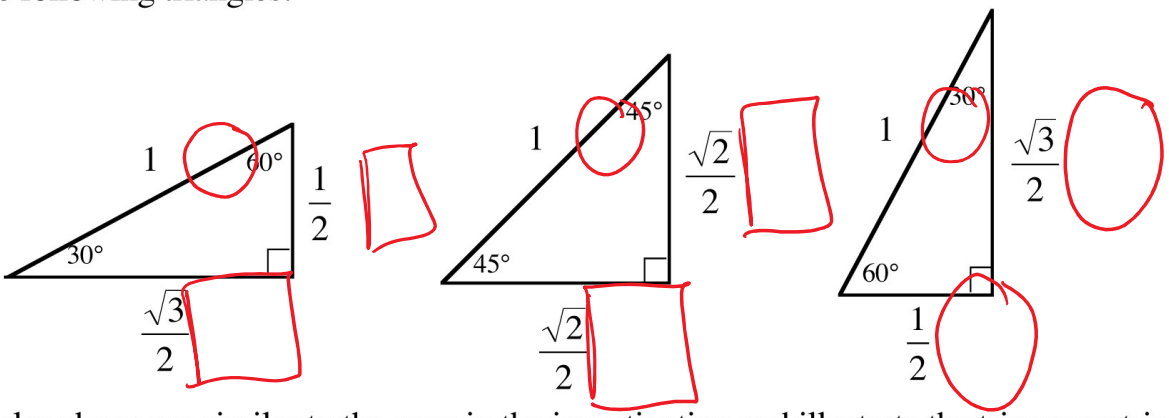
sin 30° = $\frac{y}{r}$ $= \frac{1}{2}$ $\quad$ cos 30° = $\frac{x}{r}$ $= \frac{\sqrt{3}}{2}$ $\quad$ tan 30° = $\frac{y}{x}$ $= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Special Triangles

The following triangles were developed on the previous page.



If we consider similar triangles to the above, all with hypotenuse length of one unit, we get the following triangles.



The triangles above are similar to the ones in the investigation and illustrate the trigonometric ratios as exact values for angles of 30°, 45° and 60°.

In each diagram, the horizontal distance is x, the vertical distance is y and the hypotenuse is $r = 1$.

Complete the following table.

$x°$	30°	45°	60°
sin x	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos x	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan x	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Finding Exact Trigonometric Ratios for Angles of 0° and 90°

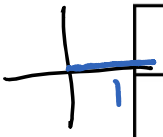
a) Consider a rotation angle of 0°.
In this case $x = 1$, $y = 0$ and $r = 1$.

b) Consider a rotation angle of 90°.
In this case $x = 0$, $y = 1$ and $r = 1$.

a) Consider a rotation angle of 0°.
In this case $x = 1$, $y = 0$ and $r = 1$.

sin 0°	$\frac{y}{r} = \frac{0}{1} = 0$
cos 0°	$\frac{x}{r} = \frac{1}{1} = 1$
tan 0°	$\frac{y}{x} = \frac{0}{1} = 0$

b) Consider a rotation angle of 90°.
In this case $x = 0$, $y = 1$ and $r = 1$.



sin 90°	$\frac{y}{r} = \frac{1}{1} = 1$
cos 90°	$\frac{x}{r} = \frac{0}{1} = 0$
tan 90°	$\frac{y}{x} = \frac{1}{0}$ undefined

c) Explain why tan 90° is undefined.

cannot divide by 0

Determining Exact Values for Trigonometric Ratios of Certain Angles

There are several ways to determine, without technology, the exact value of the sine, cosine and tangent ratios of a given angle with a reference angle of 0°, 30°, 45°, 60°, and 90°.

We will discuss two of these methods:
- by reference angle and chart
- by the unit circle (in the next lesson)

Using a Chart for Trigonometric Ratios of Special Triangles

We can summarize the exact values of trigonometric ratios of 0°, 30°, 45°, 60°, and 90° in the following chart.

x	0°	30°	45°	60°	90°
sin x	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos x	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan x	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

This chart should be memorized.

Note the following patterns:

- The sine ratios increase from 0 to 1. The cosine ratios decrease from 1 to 0.
- The tangent ratios are equal to the sine ratios divided by the cosine ratios.

Determining Exact Values of Trigonometric Ratios Using the Chart

We can use the previous table, together with the concept of reference angles and the CAST

We can use the previous table, together with the concept of reference angles and the CAST rule, to determine the exact values of the trigonometric ratios of certain angles in quadrants 2, 3, and 4.

Write the following in terms of a reference angle and determine the exact value.

a) sin 210°

- quad 3
- sine is neg.
- reference angle $210-180=30°$
- $\sin 210° = -\sin 30° = -\frac{1}{2}$ from the table

b) cos 300°

- quad 4
- cos is positive
- ref. angle $360-300$ $60°$
- $\cos 300° = \cos 60° = \frac{1}{2}$

c) tan 225°

- quad 3
- tan is positive
- ref. angle $225-180=45°$

$\tan 225° = \tan 45° = 1$

Without using technology, determine the exact value(s) of θ where

a) $\cos\theta = -\frac{\sqrt{3}}{2}$, $0° \le \theta \le 360°$

- quad 2/3
- reference angle is 30° from the table

θ = Quad 2 $180-30 = 150°$
Quad 3 $180+30 = 210°$

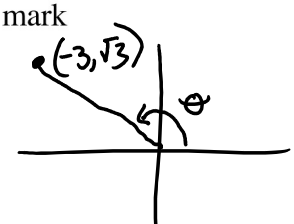
$\theta = 150°, 210°$

b) ~~tan~~ θ is undefined, $0° \le \theta \le 360$

The point $P(-3, \sqrt{3})$ (x, y) is on the terminal arm of an angle θ.

Without using technology, complete the following questions.

a) Draw the angle in standard position and mark the point P on the terminal arm.



b) State the values of x and y and hence the value of $\tan\theta$.

$x = -3$
$y = \sqrt{3}$

$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{-3}$

c) State the reference angle and hence the rotation angle θ.

c) State the reference angle and hence the rotation angle θ.

30° (from the table)

180 - 30 = 150°

#1-4, omit 3d

Complete Assignment Questions #1 - #8

Assignment

1. Complete the following chart.

x	0°	30°	45°	60°	90°
$\sin x$					
$\cos x$					
$\tan x$					

2. Find the exact value of the following using the chart and reference angle method.

a) cos 120° **b)** tan 300° **c)** sin 135° **d)** sin 330° **e)** cos 315°

f) tan 180° **g)** cos 180° **h)** sin 180° **i)** tan 150° **j)** cos 210°

k) tan 270° **l)** cos 270° **m)** sin 270° **n)** tan 240° **o)** $\cos^2 225°$

225 - 180 = 45°

$\cos^2 225 = \cos^2 45 = \left(\frac{\sqrt{2}}{2}\right)^2$

$225-180=45°$ ✓1

$\cos^2 225 = \cos^2 45 = \left(\frac{\sqrt{2}}{2}\right)^2$

$= \frac{2}{4} = \frac{1}{2}$

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3. Without using technology, determine the exact value(s) of θ where

a) $\sin\theta = \frac{\sqrt{2}}{2}$, $0° \le \theta \le 360°$ **b)** $\cos\theta = -\frac{1}{2}$, $0° \le \theta \le 360°$

c) $\tan\theta = -\frac{\sqrt{3}}{3}$, $0° \le \theta \le 360°$ **d)** $\tan\theta$ is undefined, $0° \le \theta \le 360$

4. Determine the exact measure of θ where $0° \le \theta \le 360°$.

a) $\sin\theta = 1$ **b)** $\cos\theta = 1$ **c)** $\tan\theta = 1$

d) $\sin\theta = -1$ **e)** $\cos\theta = -1$ **f)** $\tan\theta = -1$

d) $\sin\theta = -1$ **e)** $\cos\theta = -1$ **f)** $\tan\theta = -1$

g) $\sin\theta = 0$ **h)** $\cos\theta = 0$ **i)** $\tan\theta = 0$

5. The point $\left(-\sqrt{2}, -\sqrt{6}\right)$ is on the terminal arm of an angle θ.
Without using technology, complete the following questions.

a) Draw the angle in standard position and mark the point on the terminal arm.

b) State the values of x and y and hence the value of $\tan\theta$.

c) State the reference angle and hence the rotation angle θ.

6. The point (5,–5) is on the terminal arm of an angle A.
Without using technology, determine the value of angle A.

Multiple **7.** The smallest positive root of the equation $\tan x + \sqrt{3} = 0$ is

Multiple Choice **7.** The smallest positive root of the equation $\tan x + \sqrt{3} = 0$ is

A. 60° **B.** 120°
C. 150° **D.** 240°

Numerical Response **8.** The solution to the equation $2\cos\theta = -\sqrt{3}$, $180° < \theta < 360°$, to the nearest degree, is _____ .
(Record your answer in the numerical response box from left to right.)

Answer Key

1 . See table below

x	0°	30°	45°	60°	90°
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

1 $\sqrt{2}$ 1 $\sqrt{2}$

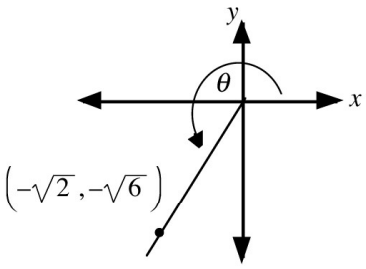
$\tan x$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

2. **a)** $-\frac{1}{2}$ **b)** $-\sqrt{3}$ **c)** $\frac{\sqrt{2}}{2}$ **d)** $-\frac{1}{2}$ **e)** $\frac{\sqrt{2}}{2}$ **f)** 0 **g)** -1 **h)** 0
i) $-\frac{\sqrt{3}}{3}$ **j)** $-\frac{\sqrt{3}}{2}$ **k)** undefined **l)** 0 **m)** -1 **n)** $\sqrt{3}$ **o)** $\frac{1}{2}$

3. **a)** 45°, 135° **b)** 120°, 240° **c)** 150°, 330° **d)** 90°, 270°

4. **a)** 90° **b)** 0°, 360° **c)** 45°, 225° **d)** 270° **e)** 180°
f) 135°, 315° **g)** 0°, 180°, 360° **h)** 90°, 270° **i)** 0°, 180°, 360°

5. **a)** see diagram below



b) $x = -\sqrt{2}$, $y = -\sqrt{6}$, $\tan\theta = \sqrt{3}$

c) reference angle = 60°, rotation angle = 240°

6. 315° **7.** B **8.**

2	1	0	