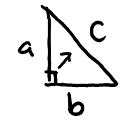
Lesson 2: Trigonometric Ratios for Angles from 0° to 360°

Trigonometry - Angles and Ratios Lesson #2:
Trigonometric Ratios for Angles from 0° to 360°

Pythagorean Theorem

$c^2 - a^2 = b^2$

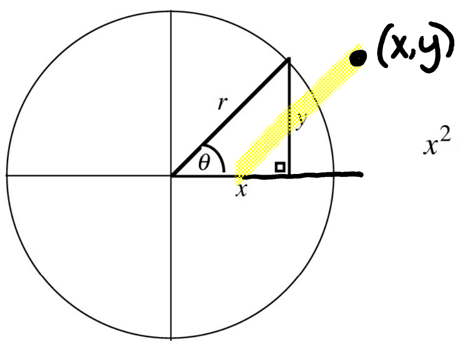


The traditional formula for the Pythagorean Theorem is $c^2 = a^2 + b^2$.

In trigonometry, we use x, y, and r instead of a, b, and c.

The point $P(x, y)$ lies on the terminal arm of angle θ.

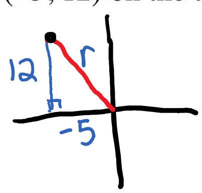
The distance from the origin to point P is r, the radius of the circle formed by the rotation.



$x^2 + y^2 = r^2$, where $r > 0$

Sketch the rotation angle in standard position, and calculate the exact distance from the origin to the given point. Where appropriate, write the answer in simplest mixed radical form.

a) Point $P(-5, 12)$ on the terminal arm of angle θ.



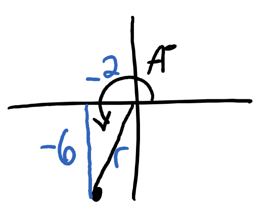
$r^2 = 12^2 + (-5)^2$

$= 144 + 25$

$\sqrt{r^2} = \sqrt{169}$

$r = 13$

b) Point $Q(-2, -6)$ on the terminal arm of angle A.



$r^2 = (-2)^2 + (-6)^2$

$r^2 = 4 + 36$

$r^2 = 40$

$r = \sqrt{40} < \sqrt{4}\ \sqrt{10} = 2\sqrt{10}$

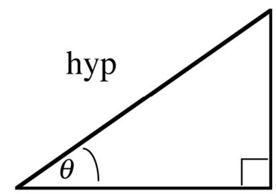
Trigonometric Ratios

Complete the following:

sine ratio ⇒ $\sin\theta = \frac{O}{H}$

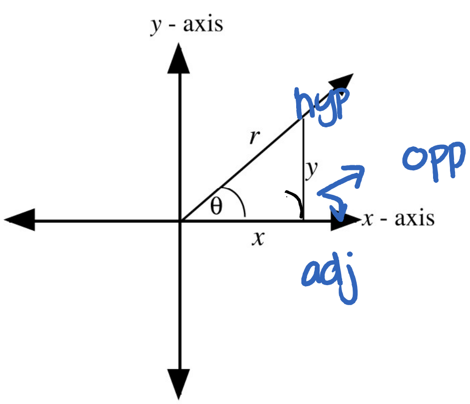
cosine ratio ⇒ $\cos\theta = \frac{A}{H}$

tangent ratio ⇒ $\tan\theta = \frac{O}{A}$



These ratios are called the ***Primary Trigonometric Ratios*** and can be remembered by the acronym **SOHCAHTOA**.

Write the primary trigonometric ratios for angle θ in terms of x, y, and r.



$\sin\theta = \frac{y}{r}$

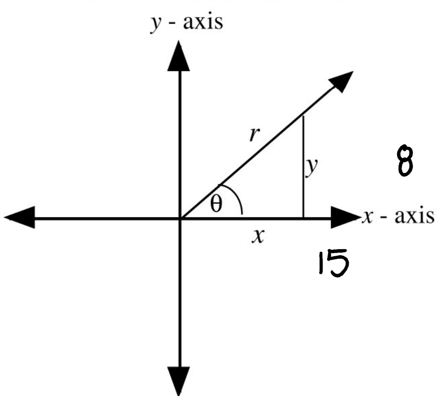
$\cos\theta = \frac{x}{r}$

$\tan\theta = \frac{y}{x}$

You should memorize these formulas.

Some students use a phrase like "**s**even **y**ellow **r**abbits" to remember **sin** $\theta = \frac{y}{r}$.

The point (15, 8) lies on the terminal arm of an angle θ as shown. Calculate the value of r, and hence determine the exact values of the primary trigonometric ratios.



$r^2 = 15^2 + 8^2$
$= 225 + 64$
$= 289$
$r = 17$

$\sin\theta = \frac{y}{r} = \frac{8}{17}$

$\cos\theta = \frac{x}{r} = \frac{15}{17}$

$\tan\theta = \frac{y}{x} = \frac{8}{15}$

$\sin\theta = 8$ $\theta = ?$

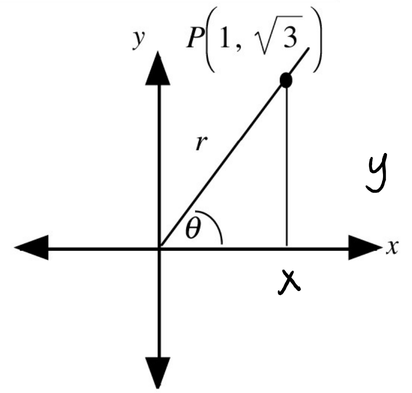
Complete Assignment Questions #1 - #5

$\sin\theta = \frac{8}{17}$ $\theta = ?$

$\theta = \sin^{-1}(8 \div 17)$

Investigating Trigonometric Ratios for Angles Between 90° and 360°

Part 1



Consider an angle θ in standard position with the point $P(1, \sqrt{3})$ on the terminal arm.

a) Show that the value of θ is 60°.

$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{1}$ $\tan\theta = \sqrt{3}$

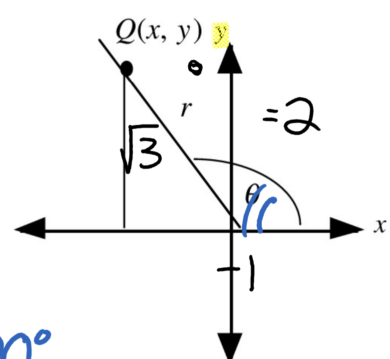
$\theta = \tan^{-1}(\sqrt{3}) = 60°$

b) Calculate the value of r. $r^2 = x^2 + y^2$ $r^2 = 1^2 + \sqrt{3}^2 = 1 + 3 = 4$ $r^2 = 4$

c) Complete the following, using $x =$ ______ , $y =$ ___$\sqrt{3}$___ and $r =$ ___2___ $r = 2$

$\sin 60° = \frac{y}{r} = \frac{\sqrt{3}}{2}$ $\cos 60° = \frac{x}{r} = \frac{1}{2}$ $\tan 60° = \frac{y}{x} = \frac{\sqrt{3}}{1} = \sqrt{3}$

Part 2



The rotation angle in Part 1 is reflected in the y-axis.

Complete the following:

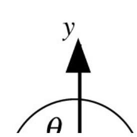
a) The point $Q(x, y)$ has coordinates $Q($ -1 , $\sqrt{3}$ $)$.

b) The reference angle is ___60°___ and the rotation angle is ___120°___.

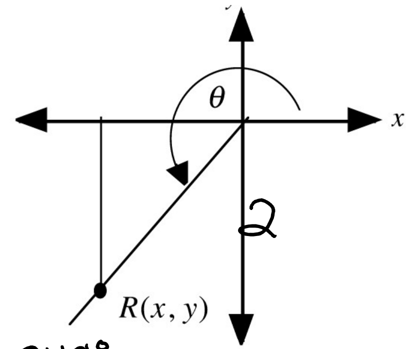
c) $\sin 120° = \frac{y}{r} = \frac{\sqrt{3}}{2}$ $\cos 120° = \frac{x}{r} = \frac{-1}{2}$ $\tan 120° = \frac{y}{x} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$

d) Confirm these trigonometric ratios on your calculator.

Part 3



Part 3



The rotation angle in Part 1 is reflected in both the x-axis and the y-axis.

Complete the following:

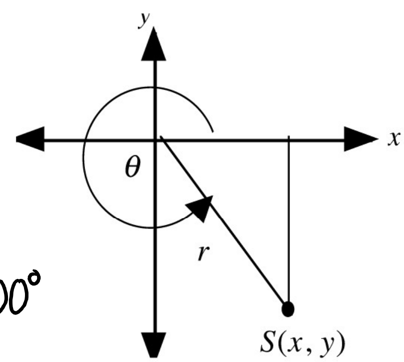
a) The point $R(x, y)$ has coordinates $R($, $)$. −1, −√3

b) The reference angle is ___60°___ and the rotation angle is ___240°___.

c) $\sin 240° = \frac{y}{r} =$ $\frac{-\sqrt{3}}{2}$ $\cos 240° = \frac{x}{r} =$ $\frac{-1}{2}$ $\tan 240° = \frac{y}{x} =$ $\frac{-\sqrt{3}}{-1} = \sqrt{3}$

d) Confirm these trigonometric ratios on your calculator.

Part 4



The rotation angle in Part 1 is reflected in the x-axis.
Complete the following:

a) The point $S(x, y)$ has coordinates $S($, $)$. 1, −√3

b) The reference angle is ___60°___ and the rotation angle is ___300°___.

c) $\sin 300° = \frac{y}{r} =$ $\frac{-\sqrt{3}}{2}$ $\cos 300° = \frac{x}{r} =$ $\frac{1}{2}$ $\tan 300° = \frac{y}{x} =$ $\frac{-\sqrt{3}}{1} = -\sqrt{3}$

d) Confirm these trigonometric ratios on your calculator.

Observations

- The trigonometric ratios for angles between 90° and 360° are either the trigonometric ratios of the reference angle, or the negative of the trigonometric ratios of the reference angle.
- The sign of the trigonometric ratios depends on the quadrant and whether x and y are positive or negative.

Determining the Sign of a Trigonometric Ratio

a) In quadrant 1, draw the rotation angle θ in standard position and complete the table.
b) Repeat for quadrants 2 - 4.

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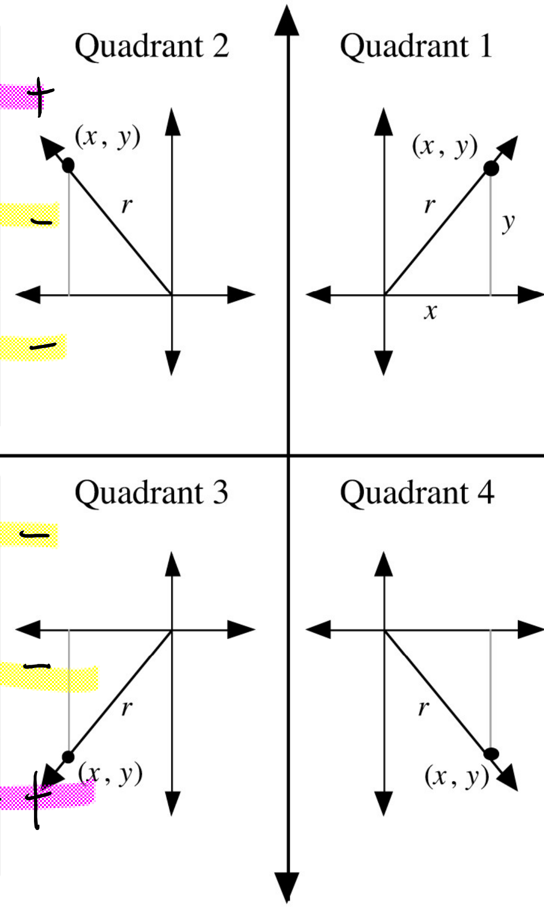
b) Repeat for quadrants 2 - 4.

Quadrant 2

$\sin\theta =$	$\frac{y}{r}$	$\frac{+}{+}$	+
$\cos\theta =$	$\frac{x}{r}$	$\frac{-}{+}$	−
$\tan\theta =$	$\frac{y}{x}$	$\frac{+}{-}$	−

Quadrant 1

$\sin\theta =$	$\frac{y}{r}$	$\frac{+}{+}$	+
$\cos\theta =$	$\frac{x}{r}$	$\frac{+}{+}$	+
$\tan\theta =$	$\frac{y}{x}$	$\frac{+}{+}$	+



Quadrant 3

$\sin\theta =$	$\frac{y}{r}$	$\frac{-}{+}$	−
$\cos\theta =$	$\frac{x}{r}$	$\frac{-}{+}$	−
$\tan\theta =$	$\frac{y}{x}$	$\frac{-}{-}$	+

Quadrant 4

$\sin\theta =$	$\frac{y}{r}$	$\frac{-}{+}$	−
$\cos\theta =$	$\frac{x}{r}$	$\frac{+}{+}$	+
$\tan\theta =$	$\frac{y}{x}$	$\frac{-}{+}$	−

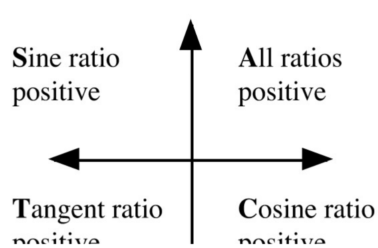
c) Complete the following statements using the results from a) and b).

i) Sine ratios have **positive** values in quadrants 1 and 2.
ii) Cosine ratios have **positive** values in quadrants 1 and 4.
iii) Tangent ratios have **positive** values in quadrants 1 and 3.
iv) Sine ratios have **negative** values in quadrants 3 and 4.
v) Cosine ratios have **negative** values in quadrants 2 and 3.
vi) Tangent ratios have **negative** values in quadrants 2 and 4.

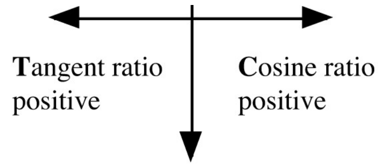
CAST Rule

The results can be memorized by:

- the **CAST** rule or
- by remembering to "**A**dd **S**ugar **T**o **C**offee"



• by remembering to "Add Sugar To Coffee"



Class Ex. #4 Determine, without using technology, whether the given trigonometric ratios are positive or negative.

a) sin 340° — Q4 neg
b) tan 227° — Q3 pos
c) sin 88° — Q1 pos

d) cos 235° — Q3 neg
e) cos 308° — Q4 pos
f) tan 123° — Q2 neg

Trigonometric Ratios of an Angle in Terms of the Reference Angle

The trigonometric ratios for any angle are either the trigonometric ratios of the reference angle, or the negative of the trigonometric ratios of the reference angle.

Use the following procedure:

i) Determine the sign of the ratio (positive or negative).
ii) Determine the measure of the reference angle.
iii) Combine i) and ii).

260-180

To write cos 260° as the cosine of an acute angle using the above procedure, we have

i) negative ii) 80° iii) cos 260° = – cos 80°.

The result can be verified on a calculator.

Class Ex. #5 Rewrite as the same trigonometric function of an acute angle.

a) sin 140° — 180 − 140; i) positive ii) 40° iii) sin 40°
b) tan 323° — 360-323; i) negative ii) 37° iii) −tan 37
c) cos 165° — 180-165; i) neg ii) 15° iii) −cos 15

d) sin 287° — 360-287; i) neg ii) 73° iii) − sin 73
e) cos 308°
f) tan 199°

iii) − sin73

Patterns in Trigonometric Ratios

We have the following pattern of results relating the trigonometric ratios of rotation angles to the trigonometric ratios of reference angles.

Let $x°$ be the reference angle for an angle in standard position.

$\sin(180 - x)° = \sin x°$ $\quad\cos(180 - x)° = -\cos x°$ $\quad\tan(180 - x)° = -\tan x°$

$\sin(180 + x)° = -\sin x°$ $\quad\cos(180 + x)° = -\cos x°$ $\quad\tan(180 + x)° = \tan x°$

$\sin(360 - x)° = -\sin x°$ $\quad\cos(360 - x)° = \cos x°$ $\quad\tan(360 - x)° = -\tan x°$

#1-8

Complete Assignment Questions #6 - #11 and the Group Investigation.

Assignment

1. Sketch the rotation angle in standard position, and calculate the exact distance from the origin to the given point.

a) Point $P(15, -8)$ on the terminal arm of angle θ.

b) Point $Q(-24, -7)$ on the terminal arm of angle B.

2. Point $P(x, y)$ is on the terminal arm of angle θ in standard position. The distance $OP = r$, where O is the origin. Express the three primary trigonometric ratios in terms of x, y, and r.

$\sin\theta =$ $\quad\cos\theta =$ $\quad\tan\theta =$

3. The point (9, 12) lies on the terminal arm of an angle θ as shown. Calculate the value of r, and hence determine the exact values of the primary trigonometric ratios.

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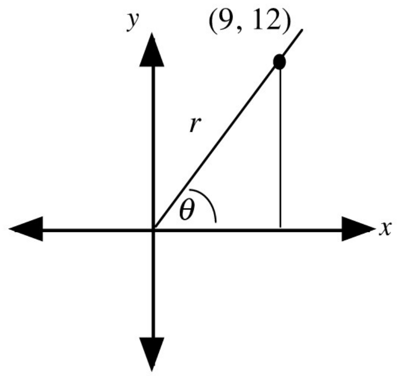


4. The point $(5, 4)$ lies on the terminal arm of an angle θ. Determine the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$. Answer as an exact radical with a rational denominator.

5. The point $(6, 12)$ lies on the terminal arm of an angle θ. Determine the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$. Answer as a mixed radical in simplest form with a rational denominator.

6. In which quadrant(s) does the terminal arm of θ lie if:

a) $\sin\theta$ is positive?

b) $\tan\theta$ is positive?

c) $\cos\theta$ is negative?

d) both $\sin\theta$ and $\tan\theta$ are negative?

e) $\cos\theta$ is positive and $\sin\theta$ is negative?

7. Determine, without using technology, whether the given trigonometric ratios are positive or negative.

a) cos 310° **b)** sin 94° **c)** tan 265°

d) sin 288° **e)** tan 109° **f)** cos 207°

8. Rewrite as the same trigonometric function of a positive acute angle.

a) sin 205° = **b)** tan 193° =

c) cos 97° = **d)** sin 156° =

e) cos 321° = **f)** tan 340° =

Multiple Choice

9. Without using technology, determine which of the following has a different sign from the others.

A. tan 255°
B. sin 272°
C. cos 175°
D. –tan 75°

10. Without using technology, determine which of the following has the same value as cos 297°.

A. cos 27°
B. cos 117°
C. –cos 243°
D. –cos 63°

Numerical Response

11. Consider angles A, B, and C such that $\cos A = \cos 217°$, $\tan B = \tan 298°$, and $\sin C = \sin 7°$, where $0° \leq A \leq 360°$, $0° \leq B \leq 360°$, and $0° \leq C \leq 360°$.

The value of $A + B + C$ is _____.

(Record your answer in the numerical response box from left to right.)

Group Investigation

The following problems are a lead in to the next lesson.

a) Sketch an angle of 30° in standard position with the point $P\left(\sqrt{3}\,, 1\right)$ on the terminal arm.

Without using technology, explain and carry out a strategy to determine the exact trigonometric ratios of three different angles greater than 90° and less than 360°.

b) Consider an angle A in standard position with $\sin A = -\frac{3}{5}$ and $0° \leq A \leq 360°$.

Without using technology, explain and carry out a strategy to determine the exact values of $\cos A$ and $\tan A$.

Answer Key

1. a) 17 **b)** 25

2. $\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{y}{x}$

3. $r = 15$, $\sin\theta = \frac{4}{5}$ $\cos\theta = \frac{3}{5}$ $\tan\theta = \frac{4}{3}$

4. $\sin\theta = \frac{4\sqrt{41}}{41}$ $\cos\theta = \frac{5\sqrt{41}}{41}$ $\tan\theta = \frac{4}{5}$

5. $\sin\theta = \frac{2\sqrt{5}}{5}$ $\cos\theta = \frac{\sqrt{5}}{5}$ $\tan\theta = 2$

6. a) 1 or 2 **b)** 1 or 3 **c)** 2 or 3 **d)** 4 **e)** 4

7. a) Positive **b)** Positive **c)** Positive **d)** Negative **e)** Negative **f)** Negative

8. a) –sin 25° **b)** tan 13° **c)** –cos 83° **d)** sin 24° **e)** cos 39° **f)** –tan 20°

9. A **10.** C **11.**

4	3	4	

Group Investigation

a) $\sin 150° = \frac{1}{2}$ $\cos 150° = -\frac{\sqrt{3}}{2}$ $\tan 150° = -\frac{\sqrt{3}}{3}$

$\sin 210° = -\frac{1}{2}$ $\cos 210° = -\frac{\sqrt{3}}{2}$ $\tan 210° = \frac{\sqrt{3}}{3}$

$\sin 330° = -\frac{1}{2}$ $\cos 330° = \frac{\sqrt{3}}{2}$ $\tan 330° = -\frac{\sqrt{3}}{3}$

b) In quadrant three, $\cos A = -\frac{4}{5}$ and $\tan A = \frac{3}{4}$.

In quadrant four, $\cos A = \frac{4}{5}$ and $\tan A = -\frac{3}{4}$.

In quadrant four, $\cos A = \frac{4}{5}$ and $\tan A = -\frac{3}{4}$.

Lesson 3: Applications of Reference Angles and the CAST Rule

Trigonometry - Angles and Ratios Lesson #3:
Applications of Reference Angles and the CAST Rule

Overview

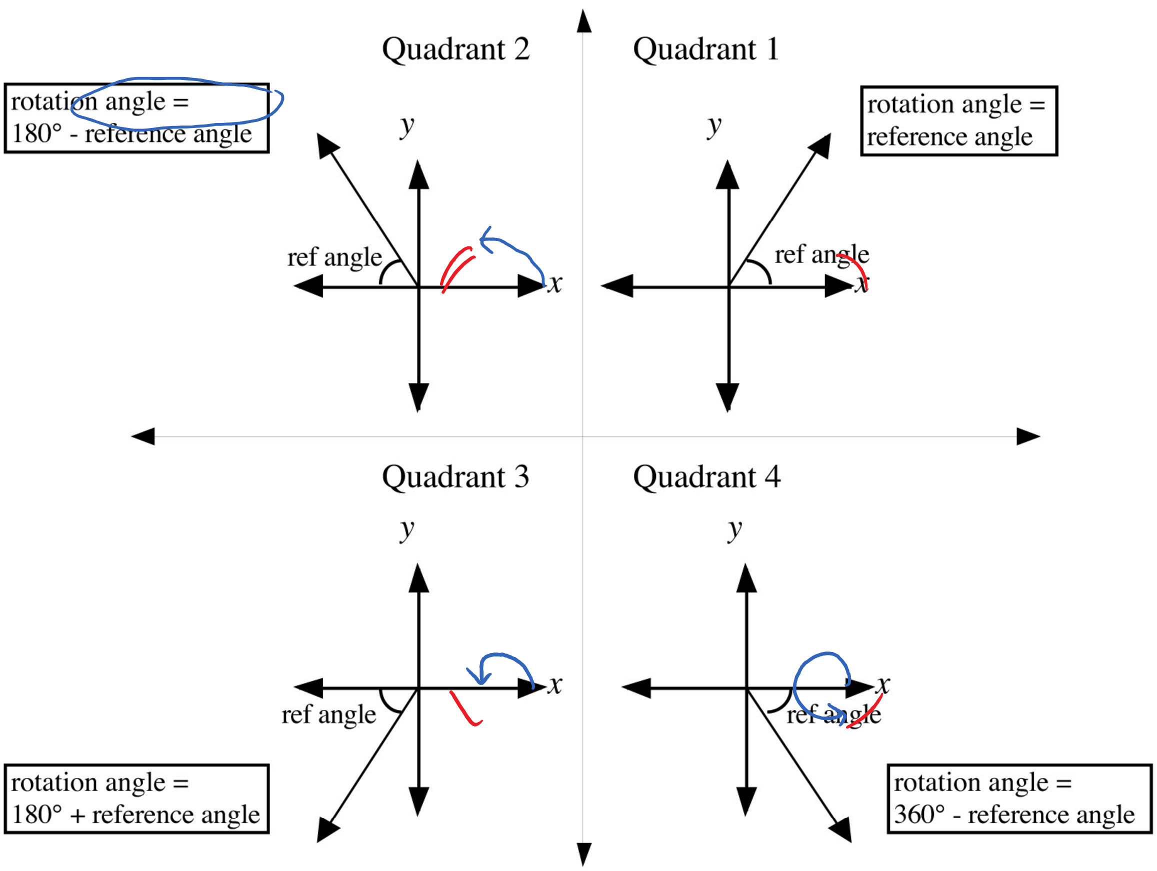
In this lesson, we use our knowledge of rotation and reference angles, and the CAST rule to:

i) determine the exact trigonometric ratios for rotation angles from 0° to 360° given a point on the terminal arm.

ii) determine trigonometric ratios for a rotation angle from 0° to 360° given a different trigonometric ratio for the angle.

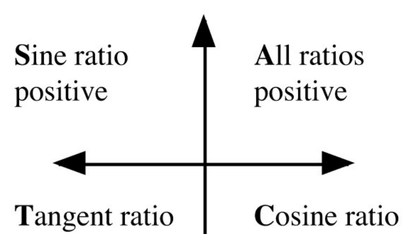
Review

The reference angle for any rotation angle is the acute angle between the terminal arm of the rotation angle and the *x*-axis.

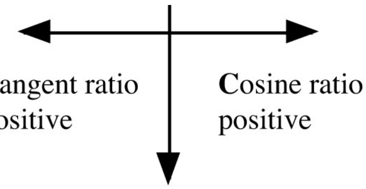


We can determine the sign of a trigonometric ratio in a particular quadrant:

- by the **CAST** rule or
- by remembering to "**A**dd **S**ugar **T**o **C**offee"



- by the **CAST** rule or
- by remembering to "**A**dd **S**ugar **T**o **C**offee"



The trigonometric ratios for an angle in standard position with a point $P(x, y)$ on the terminal arm and $OP = r$ are

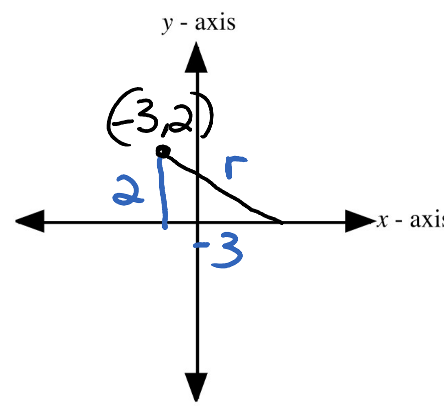
$\sin\theta = \frac{y}{r}$ $\qquad$ $\cos\theta = \frac{x}{r}$ $\qquad$ $\tan\theta = \frac{y}{x}$

Exact Values of Trigonometric Ratios Given a Point on a Terminal Arm

In the previous lesson, we were able to determine the exact values of the trigonometric ratios given a point on the terminal arm of a rotation angle in quadrant one.
In this lesson, we extend the method into quadrants two to four.

The point $P(-3, 2)$ lies on the terminal arm of an angle θ in standard position. Complete the following procedure to determine the values of the primary trigonometric ratios.

a) Sketch the rotation angle on the grid and mark the point $P(-3, 2)$ on the terminal arm.



b) Calculate the exact length of $OP = r$.

$r^2 = 2^2 + (-3)^2$
$= 4 + 9$
$= 13$
$r = \sqrt{13}$

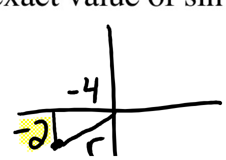
c) Use $x = -3$, $y = 2$ and r from above to write the three trigonometric ratios for angle θ.

$x = -3 \quad y = 2 \quad r = \sqrt{13}$

$\sin\theta = \frac{y}{r} = \frac{2}{\sqrt{13}} \cdot \frac{\sqrt{13}}{\sqrt{13}} = \frac{2\sqrt{13}}{13}$

$\cos\theta = \frac{x}{r} = \frac{-3}{\sqrt{13}} = \frac{-3\sqrt{13}}{13}$ $\qquad$ $\tan\theta = \frac{y}{x} = \frac{2}{-3}$

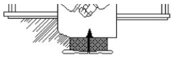
The point $(-4, -2)$ lies on the terminal arm of an angle θ in standard position. Determine the exact value of $\sin\theta$.

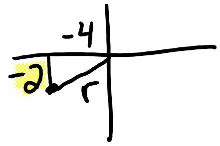


$\sin\theta = \frac{y}{r}$

① Solve for r

$\sin\theta = \frac{-2}{2\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{-2\sqrt{5}}{2 \cdot 5}$





① Solve for r

$r^2 = (-2)^2 + (-4)^2$

$= 20$

$r = \sqrt{20} = \sqrt{4 \cdot 5} = 2\sqrt{5}$

$= -\frac{\sqrt{5}}{5}$

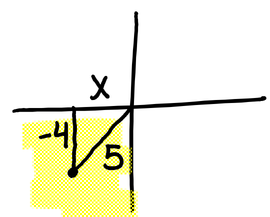
Complete Assignment Questions #1 - #3

Value of a Trigonometric Ratio Given a Different Trigonometric Ratio

Angle A terminates in the third quadrant with $\sin A = -\frac{4}{5}$. Complete the following procedure to determine the values of $\cos A$ and $\tan A$.

$\sin A = \frac{-y}{r}$

a) Since $\sin A = -\frac{4}{5} = \frac{y}{r}$, we know that the point $(x, -4)$ lies on the terminal arm in the third quadrant with $r = 5$. Sketch a diagram, draw the reference triangle and mark x, $y = -4$, and $r = 5$ on the reference triangle.



b) Use $x^2 + y^2 = r^2$ to determine the value of x. (Note that in quadrant three, the value of x must be negative).

$x^2 = r^2 - y^2$

$x^2 = 5^2 - (-4)^2$

$= 9$

$x = \pm\sqrt{9} = \pm 3$ ← x is negative in Quad 3

$x = -3$

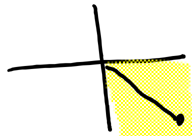
c) Use the values of x, y, and r to determine the exact values of $\cos A$ and $\tan A$.

$x = -3 \quad y = -4 \quad r = 5 \qquad \cos A = \frac{x}{r} = \frac{-3}{5}$

$\tan A = \frac{y}{x} = \frac{-4}{-3} = \frac{4}{3}$

If $\tan\theta = -\frac{2}{3}$ and $\cos\theta$ is positive, then find the exact value of $\sin\theta$.

$\ominus$ tan & $\oplus$ cos = Quad 4



← Quad 4 $\oplus x$, $\ominus y$

$\frac{y}{x} = \frac{-2}{3}$

$y = -2,\ x = 3$

$\sin\theta = \frac{y}{r}$

$r^2 = (-2) + 3^2$
$= 13$
$r = \sqrt{13}$

$\sin\theta = \frac{-2}{\sqrt{13}} = \frac{-2\sqrt{13}}{13}$

Complete Assignment Questions #4 - #11

Assignment

1. The point (8, –6) lies on the terminal arm of an angle θ in standard position. Determine the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$.

2. The point (–1, –3) lies on the terminal arm of an angle θ in standard position. Determine the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$.

3. The point (–16, 63) lies on the terminal arm of an angle A in standard position. Determine the exact value of cos A.

4. If $\cos\theta = \frac{12}{13}$ and $270° \leq \theta \leq 360°$, then find the exact values of $\sin\theta$ and $\tan\theta$.

5. If $\sin\theta = -\frac{4}{7}$ and $\cos\theta$ is negative, then find the exact value of $\tan\theta$.

6. If $\tan A = -\frac{15}{8}$ and $0° \leq A \leq 180°$, then find the values of $\sin A$ and $\cos A$.

7. If $\tan B = 0.8$ and $\cos B$ is negative, then find the exact value of $\sin B$.

8. If $\sin X = -\frac{1}{4}$ and $\tan X$ is negative, express $\cos X$ as an exact value.

9. Solve for the required ratios in each of the following. Express each answer as an exact

9. Solve for the required ratios in each of the following. Express each answer as an exact value with a rational denominator.

a) Angle θ terminates in the second quadrant. If $\tan\theta = -\frac{\sqrt{3}}{5}$, find $\sin\theta$ and $\cos\theta$.

b) Angle θ terminates in the fourth quadrant. If $\tan\theta = -\frac{\sqrt{3}}{5}$, find $\sin\theta$ and $\cos\theta$.

Multiple Choice

10. If $\cos A = -\frac{7}{25}$ and $180° \leq A \leq 270°$, then the values of $\sin A$ and $\tan A$ respectively are

A. $-\frac{24}{25}$ and $-\frac{24}{7}$

B. $-\frac{24}{25}$ and $\frac{24}{7}$

C. $-\frac{24}{25}$ and $\frac{7}{24}$

D. $\frac{24}{25}$ and $\frac{24}{7}$

11. Angle P has a terminal arm in the third quadrant. If $\tan P = \frac{1}{\sqrt{3}}$, the value of $\sin P - \cos P$ is

A. $\frac{1 - \sqrt{3}}{2}$

B. $\frac{\sqrt{3} - 1}{2}$

C. $\frac{1 + \sqrt{3}}{2}$

D. $\frac{-1 - \sqrt{3}}{2}$

Answer Key

1. $\sin\theta = -\frac{3}{5}$ $\cos\theta = \frac{4}{5}$ $\tan\theta = -\frac{3}{4}$

2. $\sin\theta = -\frac{3\sqrt{10}}{10}$ $\cos\theta = -\frac{\sqrt{10}}{10}$ $\tan\theta = 3$

1. $\sin\theta = -\frac{3}{5}$ $\cos\theta = \frac{4}{5}$ $\tan\theta = -\frac{3}{4}$

2. $\sin\theta = -\frac{3\sqrt{10}}{10}$ $\cos\theta = -\frac{\sqrt{10}}{10}$ $\tan\theta = 3$

3. $\cos A = -\frac{16}{65}$

4. $\sin\theta = -\frac{5}{13}$ $\tan\theta = -\frac{5}{12}$

5. $\tan\theta = \frac{4\sqrt{33}}{33}$

6. $\sin A = \frac{15}{17}$ $\cos A = -\frac{8}{17}$

7. $\sin B = -\frac{4\sqrt{41}}{41}$

8. $\cos X = \frac{\sqrt{15}}{4}$

9. **a)** $\sin\theta = \frac{\sqrt{21}}{14}$ $\cos\theta = -\frac{5\sqrt{7}}{14}$ **b)** $\sin\theta = -\frac{\sqrt{21}}{14}$ $\cos\theta = \frac{5\sqrt{7}}{14}$

10. B **11.** B