Lesson 5: Converting from General Form to Standard Form by Completing the Square

Quadratic Functions and Equations Lesson #5:
Converting from General Form to Standard Form by Completing the Square

Review

- The **general form** of a quadratic function has the equation $y = ax^2 + bx + c$.
- The **standard form** of a quadratic function has the equation $y = a(x - p)^2 + q$.
- Writing a function in standard form enables us to analyze the function more easily e.g. we can determine the vertex, axis of symmetry and maximum / minimum value of the function.

Completing the Square

$(x + 4)^2$ and $(x - 5)^2$ are examples of **perfect squares.**

a) Expand the following perfect squares.

$(x + 4)^2 = (x + 4)(x + 4) =$ $x^2 + 8x + 16$

$(x + 7)^2 = (x + 7)(x + 7) =$ $x^2 + 14x + 49$

$(x - 5)^2 = (x - 5)(x - 5) =$ $x^2 - 10x + 25$

$(x - 1)^2 = (x - 1)(x - 1) =$ $x^2 - 2x + 1$

$(x + a)^2 =$ $x^2 + 2ax + a^2$

$(x - a)^2 =$ $x^2 - 2ax + a^2$

b) Factor the following expressions into perfect squares.

$x^2 + 6x + 9 =$ $(x+3)^2$

$x \quad 2 \cdot 3 \cdot x \quad 3$

$x^2 + 12x + 36 =$ $(x+6)^2$

$x \quad 2 \cdot 6 \cdot x \quad 6$

$x^2 - 4x + 4 =$ $(x-2)^2$

$-2 \cdot 2 \cdot x$

$x^2 - 16x + 64 =$ $(x-8)^2$

$2 \cdot -8 \cdot x$

c) Add an appropriate constant so that the following expressions can be written as perfect squares.

$x^2 + 2x +$ ___ $= 1$ $(x+1)^2$

$x \quad 2 \cdot x \cdot \boxed{1}$

$x^2 + 18x +$ ___ $= 81$ $(x+9)^2$

$2\boxed{9}x$

$x^2 - 3x +$ ___ $= \frac{9}{4}$ $(x - \frac{3}{2})^2$

$x \quad 2\boxed{\frac{-3}{2}}x \quad (\frac{-3}{2})^2$

$\frac{2a}{2} = \frac{-3}{2}$

$x^2 - \frac{1}{4}x +$ ___ $= \frac{1}{64}$ $(x - \frac{1}{8})^2$

$2\boxed{-\frac{1}{8}}x = -\frac{1}{4}x \quad (-\frac{1}{8})^2$

$\frac{2\boxed{a}}{2} = \frac{-\frac{1}{4}}{2} = \frac{-1}{8}$

The process of adding a constant term to a quadratic expression to make it a perfect square is called **completing the square**.

To complete the square of $x^2 + bx$, add $\left(\frac{1}{2}\text{coefficient of } x\right)^2$

called **completing the square**.

To complete the square of $x^2 + bx$, add $\left(\frac{1}{2}\text{coefficient of } x\right)^2$

i.e. add $\left(\frac{1}{2}b\right)^2$ to give $\left(x + \frac{1}{2}b\right)^2$.

Writing $f(x) = x^2 + bx + c$ in Standard Form by Completing the Square

Use the following process to convert a function of the form $f(x) = x^2 + bx + c$ into standard form.

Step 1

Take half of the numerical coefficient of the x-term and square it.

bx

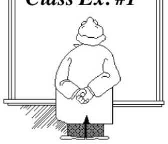
Step 2

Add and subtract the squared number in step 1. (This keeps the value of the function the same.)

Step 3

Form a perfect square, write it in factored form, and simplify.

Class Ex. #1



Express $y = x^2 + 10x + 16$ in completed square form.
Use a graphing calculator to verify that both equations are represented by identical graphs.

$$y = x^2 + 10x + 16$$
$$= x^2 + 10x + 25 - 25 + 16$$
$$= (x+5)^2 - 9$$

$\frac{1}{2}(10) = 5$

$5^2 = 25$

Class Ex. #2

A function, f, is defined by $f(x) = x^2 - 9x - 20$.
Determine the minimum value of f by writing the function in standard form.

vertex f

$-9\left(\frac{1}{2}\right) = \left(\frac{-9}{2}\right)^2 = \frac{81}{4}$

$$f(x) = x^2 - 9x + \frac{81}{4} - \frac{81}{4} - 20$$
$$= \left(x - \frac{9}{2}\right)^2 - \frac{81}{4} - \cancel{20}\,\frac{80}{4}$$
$$= \left(x - \frac{9}{2}\right)^2 - \frac{161}{4}$$

minimum value is $\frac{-161}{4}$

$= \left(x - \frac{1}{2}\right)^2 - \frac{161}{4}$

p q

minimum value is $\frac{-161}{4}$

1, 2abc

$\boxed{\text{3ace}}$ $\boxed{9:15}$

Complete Assignment Questions #1 - #4

Writing $f(x) = ax^2 + bx + c$ in Standard Form by Completing the Square

Step 1

Use the coefficient of x^2 as a common factor for the x^2 term and the x term only.

Step 2

Take half of the numerical coefficient of the x-term and square it.

Step 3

Add and subtract the squared number in step 2 inside the brackets.

Step 4

Multiply the term which has been subtracted in step 3 by the coefficient of x^2 and combine with the constant term.

Step 5

Form a perfect square within the brackets and write it in factored form.

Convert $f(x) = 3x^2 - 18x + 20$ to standard form by completing the square. Determine whether the graph of the function f has a maximum or minimum value and state the value.

$= 3x^2 - 18x \quad + 20$

$= 3(x^2 - 6x + 9 - 9) + 20$

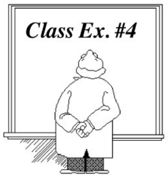
$= 3(x - 3)^2 - 27 + 20$

$= 3(x - 3)^2 - 7$

$-6\left(\frac{1}{2}\right) = (-3)^2 = 9$

minimum value because "a" is positive

min = -7



Convert $y = 7 + 10x - 2x^2$ to standard form by completing the square. In what direction does the parabola open? What are the coordinates of the vertex of the parabola?

$y = -2x^2 + 10x + 7$

$= -2\left(x^2 - 5x + \frac{25}{4} - \frac{25}{4}\right) + 7$

$= -2\left(x - \frac{5}{2}\right)^2 + \frac{50}{4} + 7 \rightarrow \frac{25}{2} + \frac{14}{2}$

$-5\left(\frac{1}{2}\right) = \left(\frac{-5}{2}\right)^2 = \frac{25}{4}$

$\boxed{y = -2\left(x - \frac{5}{2}\right)^2 + \frac{39}{2}}$

vertex $\left(\frac{5}{2}, \frac{39}{2}\right)$

opens down

Complete Assignment Questions #5 - #9

3ace, 4, 5ace

Assignment

1. What number must be added to each to make a perfect square?

a) $x^2 + 8x$ **b)** $x^2 - 24x$ **c)** $x^2 + 40x$ **d)** $x^2 - x$ **e)** $x^2 + \frac{1}{2}x$ **f)** $x^2 - \frac{2}{3}x$

2. Complete the square in each part.

a) $x^2 + 6x + \_\_ = (x + \_\_)^2$ **b)** $x^2 - 20x + \_\_ = (x \quad \_\_)^2$

c) $x^2 + 5x + \_\_ = (x \quad \_\_)^2$ **d)** $x^2 - 9x + \_\_ = (x \quad \_\_)^2$

e) $x^2 + 0.6x + \_\_ = (x \quad \_\_)^2$ **f)** $x^2 - \frac{3}{4}x + \_\_ = (x \quad \_\_)^2$

3. Express the following in completed square form.

a) $y = x^2 + 10x + 3$ **b)** $y = x^2 - 4x - 21$ **c)** $y = x^2 + 14x - 2$

d) $f(x) = x^2 + 9x + 22$ **e)** $g(x) = x^2 - x + 1$ **f)** $h(x) = x^2 + bx + c$

4. Express $f(x) = x^2 - 14x - 40$ in completed square form. Hence state the coordinates of the vertex and the equation of the axis of symmetry of the graph of the function.

5. Express the following in completed square form.

a) $f(x) = 2x^2 + 12x + 5$ **b)** $y = 3x^2 - 18x - 19$ **c)** $P(x) = 2x^2 + 14x - 11$

d) $y = -x^2 + 10x + 20$ **e)** $y = -4x^2 - 8x + 7$ **f)** $g(x) = 11x - x^2$

Multiple Choice **6.** When $y = 2x^2 + 5x + 10$ is converted to the form $y = a(x - p)^2 + q$, the value of q is

A. -2.5
B. 3.75
C. 6.875
D. 8.4375

7. The x-coordinate of the vertex of the graph of the function $f(x) = bx - 4x^2$ is

A. $\frac{b}{4}$
B. $\frac{b}{8}$
C. $\frac{b}{16}$
D. $\frac{b^2}{16}$

8. A high school student was asked to arrange the equation $y = -3x^2 - 6x - 5$ in the form $y = a(x - p)^2 + q$ by completing the square. The student's procedure is shown:

Step I: $y = -3(x^2 + 2x \qquad) - 5$
Step II: $y = -3(x^2 + 2x + 1 - 1) - 5$
Step III: $y = -3(x + 1)^2 - 5 - 1$
Step IV: $y = -3(x + 1)^2 - 6$

The student made an error in

A. Step I
B. Step II
C. Step III
D. Step IV

Numerical **9.** The maximum value, to the nearest tenth, of the function $g(x) = -5x^2 + 10x + 12$ is

Numerical Response **9.** The maximum value, to the nearest tenth, of the function $g(x) = -5x^2 + 10x + 12$ is _____

(Record your answer in the numerical response box from left to right.)

Answer Key

1. **a)** 16 **b)** 144 **c)** 400 **d)** $\frac{1}{4}$ **e)** $\frac{1}{16}$ **f)** $\frac{1}{9}$

2. **a)** $x^2 + 6x + 9 = (x+3)^2$ **b)** $x^2 - 20x + 100 = (x-10)^2$ **c)** $x^2 + 5x + \frac{25}{4} = \left(x + \frac{5}{2}\right)^2$

d) $x^2 - 9x + \frac{81}{4} = \left(x - \frac{9}{2}\right)^2$ **e)** $x^2 + 0.6 + 0.09 = (x + 0.3)^2$ **f)** $x^2 - \frac{3}{4}x + \frac{9}{64} = \left(x - \frac{3}{8}\right)^2$

3. **a)** $y = (x+5)^2 - 22$ **b)** $y = (x-2)^2 - 25$ **c)** $y = (x+7)^2 - 51$

d) $f(x) = \left(x + \frac{9}{2}\right)^2 + \frac{7}{4}$ **e)** $g(x) = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$ **f)** $g(x) = \left(x + \frac{b}{2}\right)^2 + c - \frac{b^2}{4}$

4. $(7, -89)$, $x = 7$

5. **a)** $f(x) = 2(x+3)^2 - 13$ **b)** $y = 3(x-3)^2 - 46$ **c)** $P(x) = 2\left(x + \frac{7}{2}\right)^2 - \frac{71}{2}$

d) $y = -(x-5)^2 + 45$ **e)** $y = -4(x+1)^2 + 11$ **f)** $g(x) = -\left(x - \frac{11}{2}\right)^2 + \frac{121}{4}$

6. C **7.** B **8.** C **9.**

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