Quadratic Functions and Equations Lesson #3:
Analyzing Quadratic Functions - Part Two

$y = a(x^2 - p) + q$

todays lesson

vertex (p, q)

In the last lesson we analyzed the graph of $y = (x - p)^2 + q$ and discovered transformations associated with the parameters p and q. In this lesson we investigate the effect of the parameter, a, on the graph of $y = a(x - p)^2 + q$. The following investigations can be completed as a class lesson or as an individual assignment.

Analyzing the Graph of $y = a(x - p)^2, a > 0$

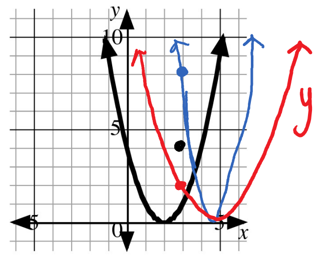
The graph of $y = f(x) = (x - 2)^2$ is shown.

$y_1 = (x-2)^2$

a) Write an equation which represents each of the following:

- $y = 2f(x)$ $\quad y_2 = 2(x-2)^2$
- $y = \frac{1}{2}f(x)$ $\quad y_3 = \frac{1}{2}(x-2)^2$ (0.5)

b) Use a graphing calculator to sketch $y = 2f(x)$ and $y = \frac{1}{2}f(x)$ on the grid.



$y = \frac{1}{2}(x-2)^2$

c) Complete the following by circling the correct choice and filling in the blank.

- Compared to the graph of $y = f(x)$, the number 2 in the graph of $y = 2f(x)$ results in a vertical *expansion / compression* by a factor of 2.
- The y intercept of the graph of $y = 2f(x)$ is double the y-intercept of the graph of $y = f(x)$.

d) Complete the following by circling the correct choice and filling in the blank.

- Compared to the graph of $y = f(x)$, the number $\frac{1}{2}$ in the graph of $y = \frac{1}{2}f(x)$ results in a vertical *expansion / compression* by a factor of $\frac{1}{2}$.
- The y intercept of the graph of $y = \frac{1}{2}f(x)$ is half the y-intercept of the graph of $y = f(x)$.

Note

- In mathematics, the general name given to an expansion or a compression is a **stretch**.
- A vertical stretch is "anchored" by the x-axis, i.e. the x-coordinate of every point on the original graph will not change and the y-coordinate of every point is multiplied by a factor of a.
- In some texts a compression is called a contraction.

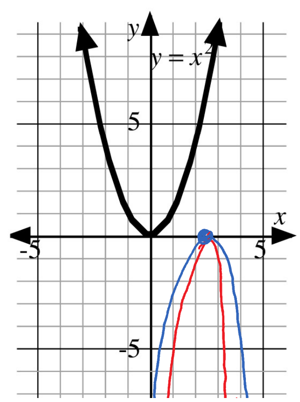
e) Describe the effect of the **parameter**, a, on the graph of $y = a(x - p)^2$ where $a > 0$. +

vertical stretch by a factor of a

f) Compared to the graph of $y = x^2$, the graph of $y = ax^2$ results in a vertical stretch of factor a about the x-axis.
If $a > 1$, the parabola undergoes a vertical expansion.
If $0 < a < 1$, the parabola undergoes a vertical compression.
fraction

Analyzing the Graph of $y = ax^2, a < 0$

The graph of $y = f(x) = x^2$ is shown. $\quad y_1 = x^2$



a) Write an equation which represents

- $y = -f(x)$ $\quad y_2 = -x^2$
- $y = -2f(x)$ $\quad y_3 = -2x^2$

b) Use a graphing calculator to sketch $y = -f(x)$ and $y = -2f(x)$.

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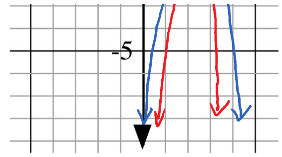


c) Complete the following chart. The first row is done.

Function	*Equation Representing Function*	*Vertex*	*Max/Min Value*	*Equation of Axis of Symmetry*	*Description of Transformation*
$y = f(x)$	$y = x^2$	(0, 0)	min, 0	$x = 0$	no transformation
$y = -f(x)$	$y = -x^2$	(0,0)	max, 0	$x = 0$	reflection in x-axis
$y = -2f(x)$	$y = -2x^2$	(0,0)	max, 0	$x = 0$	-reflection in x-axis -vertical stretch by factor of 2
$y = af(x)$, where $a < 0$	$y = ax^2$ $a < 0$	(0,0)	max, 0	$x = 0$	-reflection in x-axis -vertical stretch of a factor a

d) How does the graph of $y = -x^2$ compare to the graph of $y = x^2$?

reflection in the x-axis

e) Compared to the graph of $y = x^2$, the graph of $y = ax^2, a < 0$ results in a reflection in the x-axis and a vertical stretch by a factor of $-a$ about the x-axis.

Transformations Associated with the Parameters of $y = a(x - p)^2 + q$

Compared to the graph of $y = x^2$, the following transformations are associated with the parameters of $y = a(x - p)^2 + q$.

a indicates a vertical stretch about the x-axis.

- If $a > 1$ there is an expansion.
- If $0 < a < 1$ there is a compression. (fraction less than 1)
- If $a < 0$, there is also a reflection in the x-axis.

p indicates a horizontal translation. If $p > 0$, the parabola moves p units right. If $p < 0$, the parabola moves p units left.

q indicates a vertical translation. If $q > 0$, the parabola moves q units up. If $q < 0$, the parabola moves q units down.

(p, q) are the coordinates of the vertex.

$x = p$ is the equation of the axis of symmetry.

Class Ex. #1

Consider the function $f(x) = 2(x + 4)^2 - 3$. $y = (x-2)^2 \Rightarrow x^2 - 4x + 4$

a) State the transformations applied to the graph of $y = x^2$ which would result in the graph of $y = 2(x + 4)^2 - 3$.   vertical stretch by a factor of 2

Class Ex. #1

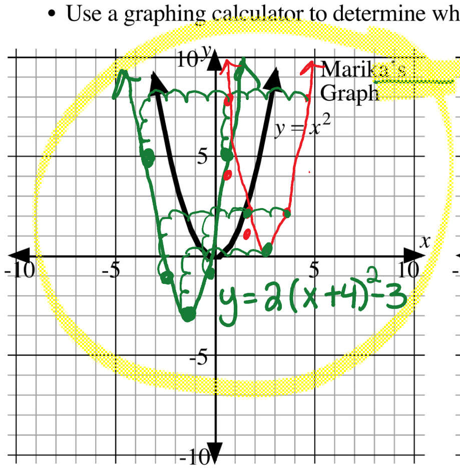
Consider the function $f(x) = 2(x + 4)^2 - 3$. $y=(x-2)^2 \rightarrow x^2-4x+4$

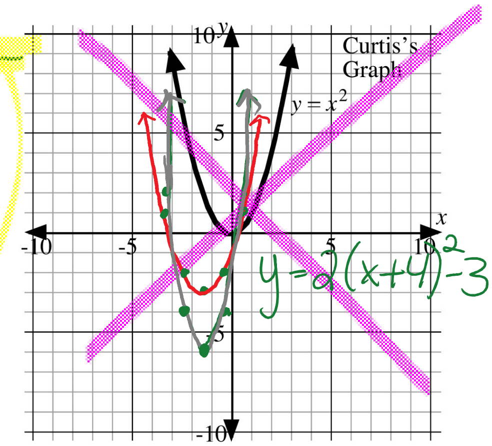
a) State the transformations applied to the graph of $y = x^2$ which would result in the graph of $y = 2(x + 4)^2 - 3$.

vertical stretch by a factor of 2
$p=-4$, 4 units to the left
$q=-3$, shift 3 units down

b) Marika and Curtis were discussing how to graph this function without using a graphing calculator. Marika suggested doing the stretch followed by the translation. Curtis suggested doing the translation followed by the stretch. shift

- Complete the grids below to show the graphs obtained by each student.
- Use a graphing calculator to determine which student is correct.





Note Unless otherwise indicated, use the following order to describe how to transform from one graph to another.

1. Stretches
2. Reflections
3. Translations

} this order matters!

Class Ex. #2

Describe how the graphs of the following functions relate to the graph of $y = x^2$.

a) $y = -\frac{1}{4}x^2$

- vertical stretch by a factor of $\frac{1}{4}$
- reflection in the x-axis

b) $\frac{1}{3}y = (x + 6)^2$ $\rightarrow y = 3(x+6)^2$

- vertical stretch of 3
- $p=-6$, shift 6 units left

Class Ex. #3

The following three transformations are applied, in order, to the graph of $y = x^2$: a reflection in the x-axis, a vertical stretch by a factor of $\frac{1}{3}$ about the x-axis, and a translation 7 units right. At the end of the three transformations, the point $(1, t)$ is on the resulting graph. $p=7$

a) Find the equation of the image function after each transformation.

$x^2 \rightarrow -x^2 \rightarrow -\frac{1}{3}x^2 \rightarrow \boxed{-\frac{1}{3}(x-7)^2} + 0$

b) State the coordinates of the vertex (p, q) of the final graph. $(7,0)$

c) Find the value of t.

$(1, t)$
(x, y)

$y = -\frac{1}{3}(x-7)^2$ solve for y

$= -\frac{1}{3}(1-7)^2 = -\frac{1}{3}(-6)^2 = -\frac{1}{3}(36) \boxed{= -12}$

Class Ex. #4

Complete the following table.

			Equation of		

Complete the following table.

Function	Vertex	Max/Min Value	Equation of Axis of Symmetry	Domain	Range
$y = -(x + 3)^2 - 4$	$(-3,-4)$	max, -4	$x = -3$	$x \in R$	$y \leq -4$
$y = 3(x - 9)^2$	$(9,0)$	min 0	$x = 9$	$x \in R$	$y \geq 0$

(p.9)

Complete Assignment Questions #1 - #10   #1-7 $\boxed{\text{L6 \#1, 5, 6, 7, 8ac, 9ab, 10}}$ /15

Assignment

1. Describe how the graphs of the following functions relate to the graph of $y = x^2$.

a) $y = -3x^2$

b) $y = x^2 - 15$

c) $y = -\frac{2}{3}(x + 4)^2 - 1$

d) $2y = (x - 8)^2 + 12$

2. The following transformations are applied to the graph of $y = x^2$ in the order given. Write the equation of the image function for each.

a) A reflection in the x-axis and a vertical stretch by a factor of 4 about the x-axis.

b) A vertical stretch by a factor of $\frac{3}{5}$ about the x-axis, and a translation of 5 units down.

c) A vertical stretch by a factor of 8 about the x-axis, a reflection in the x-axis, a vertical translation of 3 units up, and a horizontal translation 9 units left.

d) A vertical stretch by a factor of c about the x-axis, a reflection in the x-axis, and a translation of e units right and f units down.

3. Complete the following table.

Function	Vertex	Max/Min Value	Equation of Axis of Symmetry	Domain	Range
$y = 3x^2$					
$y = 2x^2 + 1$					
$y = -(x + 7)^2$					
$y - 10 = (x + 5)^2$					
$y + 3 = -3(x - 1)^2 + 2$					

$y + 3 = -3(x - 1)^2 + 2$					

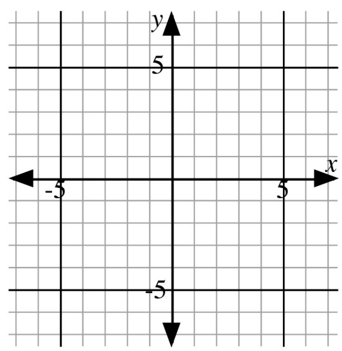
4. The following transformations are applied, **in order**, to the graph of $y = x^2$.
- A reflection in the x-axis.
- A vertical stretch of factor 3 about the x-axis.
- A translation of 5 units right and 2 units down.

a) Find the equation of the image function after each transformation.

b) At the end of all the transformations, the point $(4, y)$ is on the final graph of the parabola. Find the y-coordinate for the final graph when $x = 4$.

5. The graph of $f(x) = x^2$ undergoes a series of transformations.

a) State the transformations applied to the graph of $f(x) = x^2$ which would result in the graph of $f(x) = -\frac{1}{2}(x - 2)^2 + 1$.



b) Without using a graphing calculator, sketch the graph of $f(x) = -\frac{1}{2}(x - 2)^2 + 1$.

c) Verify using a graphing calculator.

6. Write the coordinates of the image of the point $(-3, 9)$ on the graph $y = x^2$ when each of the following transformations are applied.

a) A reflection in the x-axis, followed by a vertical translation of 4 units up.

b) A vertical stretch by a factor of $\frac{1}{3}$ about the x-axis.

7. Write the equation of a quadratic function which is the image of $y = x^2$ after a vertical stretch about the x-axis by the given factor of a, and after a translation which results in the given vertex.

a) $a = 3$, vertex $(4, -1)$

b) $a = \frac{1}{2}$, vertex $(-3, 2)$

c) $a = -4$, vertex $(0, 5)$

d) $a = -\frac{1}{3}$, vertex $(-6, -3)$

Multiple Choice **8.** The quadratic function $f(x) = x^2$ is transformed to $f(x) = -\frac{1}{2}(x + 3)^2 + 1$.

The point $(1, 1)$ on the graph of $y = x^2$ is transformed to which point on the graph of $y = -\frac{1}{2}(x + 3)^2 + 1$?

The point (1, 1) on the graph of $y = x^2$ is transformed to which point on the graph of $y = -\frac{1}{2}(x+3)^2 + 1$?

A. $\left(-2, \frac{1}{2}\right)$

B. $\left(-2, \frac{3}{2}\right)$

C. $(-2, -1)$

D. $\left(\frac{5}{2}, 2\right)$

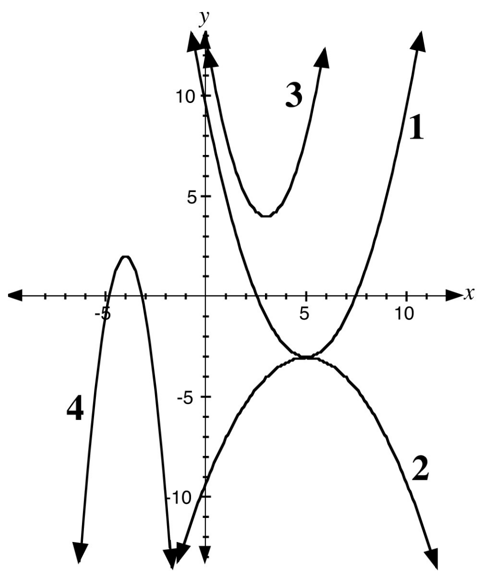
Numerical Response **9.** The diagram shows the graphs of four quadratic functions.

In the first box, write the number corresponding to the graph of $y = \frac{1}{2}(x-5)^2 - 3$.

In the second box, write the number corresponding to the graph of $y = -3(x+4)^2 + 2$.

In the third box, write the number corresponding to the graph of $y = (x-3)^2 + 4$.

In the last box, write the number corresponding to the graph of $y + 3 = -\frac{1}{4}(x-5)^2$.



(Record your answer in the numerical response box from left to right.)

10. The following transformations are applied, in order, to the graph of $y = x^2$.
- A vertical stretch of factor 2 about the x-axis.
- A reflection in the x-axis.
- A vertical translation of 12 units up.

At the end of all the transformations, the point $(2, y)$ is on the final graph of the parabola. The value of y, to the nearest tenth, is _____ .

(Record your answer in the numerical response box from left to right.)

Answer Key

1. a) vertical stretch by a factor of 3 about the x-axis and a reflection in the x-axis

b) vertical translation 15 units down

c) vertical stretch by a factor of $\frac{2}{3}$ about the x-axis, a reflection in the x-axis, a translation 4 units left, 1 unit down

d) vertical stretch by a factor of $\frac{1}{2}$ about the x-axis, and a translation 8 units right, 6 units up

2. a) $y = -4x^2$ **b)** $y = \frac{3}{5}x^2 - 5$ **c)** $y = -8(x+9)^2 + 3$ **d)** $y = -c(x-e)^2 - f$

3.

Function	Vertex	Max/Min Value	Equation of Axis of Symmetry	Domain	Range
$y = 3x^2$	$(0, 0)$	min, 0	$x = 0$	$\{x \mid x \in \Re\}$	$\{y \mid y \geq 0, y \in \Re\}$
$y = 2x^2 + 1$	$(0, 1)$	min, 1	$x = 0$	$\{x \mid x \in \Re\}$	$\{y \mid y \geq 1, y \in \Re\}$

		Value	Symmetry		
$y = 3x^2$	$(0, 0)$	min, 0	$x = 0$	$\{x \mid x \in \Re\}$	$\{y \mid y \geq 0, y \in \Re\}$
$y = 2x^2 + 1$	$(0, 1)$	min, 1	$x = 0$	$\{x \mid x \in \Re\}$	$\{y \mid y \geq 1, y \in \Re\}$
$y = -(x + 7)^2$	$(-7, 0)$	max, 0	$x = -7$	$\{x \mid x \in \Re\}$	$\{y \mid y \leq 0, y \in \Re\}$
$y - 10 = (x + 5)^2$	$(-5, 10)$	min, 10	$x = -5$	$\{x \mid x \in \Re\}$	$\{y \mid y \geq 10, y \in \Re\}$
$y + 3 = -3(x - 1)^2 + 2$	$(1, -1)$	max, -1	$x = 1$	$\{x \mid x \in \Re\}$	$\{y \mid y \leq -1, y \in \Re\}$

4. a) $y = -x^2,\ y = -3x^2,\ y = -3(x - 5)^2 - 2$ **b)** -5

5. a) vertical stretch by a factor of $\frac{1}{2}$ about the x-axis, a reflection in the x-axis, a translation 2 units right, 1 unit up

6. a) $(-3, -5)$ **b)** $(-3, 3)$

7. a) $y = 3(x - 4)^2 - 1$ **b)** $y = \frac{1}{2}(x + 3)^2 + 2$ **c)** $y = -4x^2 + 5$ **d)** $y = -\frac{1}{3}(x + 6)^2 - 3$

8. A

9.

1	4	3	2

10.

4	.	0	