Lesson 3: Applications of Reference Angles and the CAST Rule

Trigonometry - Angles and Ratios Lesson #3:
Applications of Reference Angles and the CAST Rule

Overview

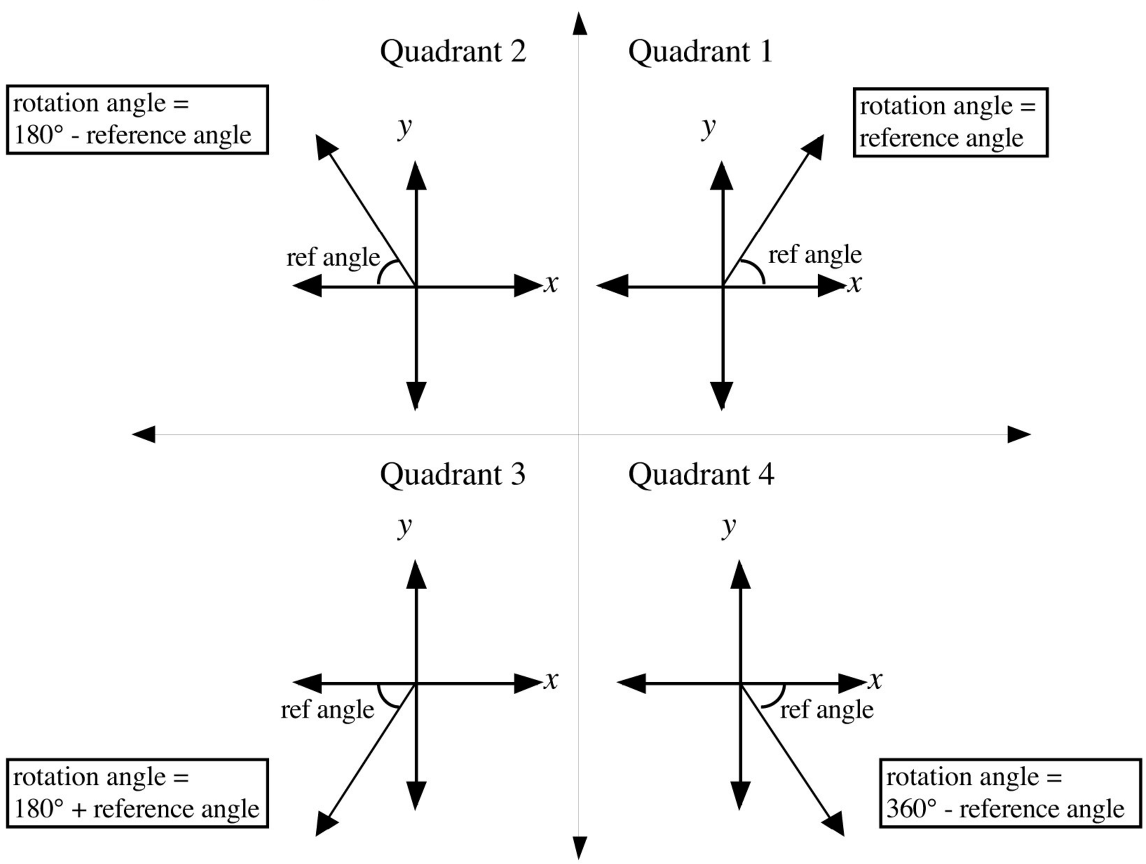
In this lesson, we use our knowledge of rotation and reference angles, and the CAST rule to:

i) determine the exact trigonometric ratios for rotation angles from 0° to 360° given a point on the terminal arm.

ii) determine trigonometric ratios for a rotation angle from 0° to 360° given a different trigonometric ratio for the angle.

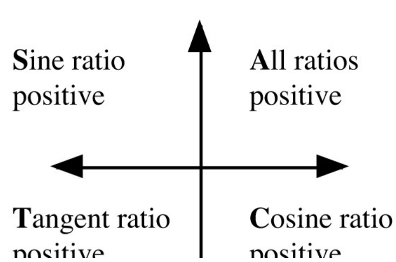
Review

The reference angle for any rotation angle is the acute angle between the terminal arm of the rotation angle and the *x*-axis.



We can determine the sign of a trigonometric ratio in a particular quadrant:

- by the **CAST** rule or
- by remembering to "**A**dd **S**ugar **T**o **C**offee"



- by the **CAST** rule or
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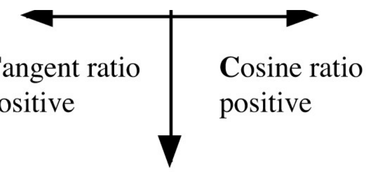


The trigonometric ratios for an angle in standard position with a point $P(x, y)$ on the terminal arm and $OP = r$ are

$\sin\theta = \frac{y}{r}$ $\qquad\cos\theta = \frac{x}{r}$ $\qquad\tan\theta = \frac{y}{x}$

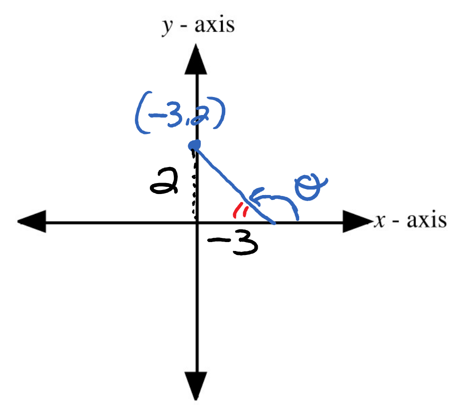
Exact Values of Trigonometric Ratios Given a Point on a Terminal Arm

In the previous lesson, we were able to determine the exact values of the trigonometric ratios given a point on the terminal arm of a rotation angle in quadrant one.
In this lesson, we extend the method into quadrants two to four.

Class Ex. #1

The point $P(-3, 2)$ lies on the terminal arm of an angle θ in standard position. Complete the following procedure to determine the values of the primary trigonometric ratios.

a) Sketch the rotation angle on the grid and mark the point $P(-3, 2)$ on the terminal arm.



b) Calculate the exact length of $OP = r$.

$r^2 = 2^2 + (-3)^2$
$= 4 + 9$
$r^2 = 13$
$r = \sqrt{13}$

c) Use $x = -3$, $y = 2$ and r from above to write the three trigonometric ratios for angle θ.

$\sin\theta = \frac{y}{r} = \frac{2}{\sqrt{13}}$ or $\frac{2\sqrt{13}}{13}$

$\cos\theta = \frac{x}{r} = \frac{-3}{\sqrt{13}}$ or $\frac{-3\sqrt{13}}{13}$

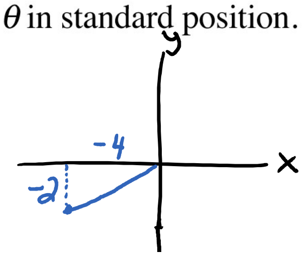
$\tan\theta = \frac{y}{x} = \frac{2}{-3}$

Class Ex. #2

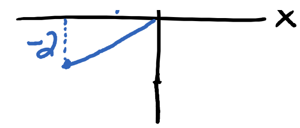
The point $(-4, -2)$ lies on the terminal arm of an angle θ in standard position. Determine the exact value of $\sin\theta$.

$x = -4$, $y = -2$ $\qquad r = ?$

$r^2 = (-4)^2 + (-2)^2$
$= 16 + 4$



$r^2 = (-4)^2 + (-2)^2$
$= 16 + 4$
$r = \sqrt{20} = 2\sqrt{5}$



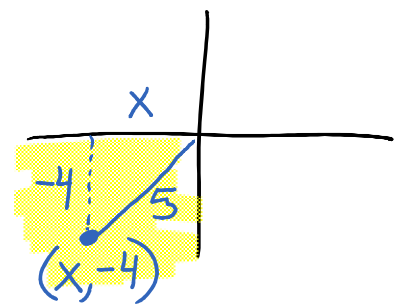
$\sin\theta = \frac{y}{r} = \frac{-2}{2\sqrt{5}} = \frac{-1}{\sqrt{5}} = \frac{-\sqrt{5}}{5}$

Complete Assignment Questions #1 - #3

Value of a Trigonometric Ratio Given a Different Trigonometric Ratio

Angle A terminates in the third quadrant with $\sin A = -\frac{4}{5}$. Complete the following procedure to determine the values of $\cos A$ and $\tan A$.

a) Since $\sin A = -\frac{4}{5} = \frac{y}{r}$, we know that the point $(x, -4)$ lies on the terminal arm in the third quadrant with $r = 5$. Sketch a diagram, draw the reference triangle and mark x, $y = -4$, and $r = 5$ on the reference triangle.



b) Use $x^2 + y^2 = r^2$ to determine the value of x.
(Note that in quadrant three, the value of x must be negative).

$x^2 = r^2 - y^2$
$x^2 = 5^2 - (-4)^2$
$x^2 = 25 - 16$
$x^2 = 9$
$x = \sqrt{9} = \pm 3$
$x = -3$

c) Use the values of x, y, and r to determine the exact values of $\cos A$ and $\tan A$.

$x = -3 \quad y = -4 \quad r = 5$

$\cos A = \frac{x}{r} = \frac{-3}{5}$

$\tan A = \frac{y}{x} = \frac{-4}{-3} = \frac{4}{3}$

If $\tan\theta = -\frac{2}{3}$ and $\cos\theta$ is positive, then find the exact value of $\sin\theta$.

Quad 4

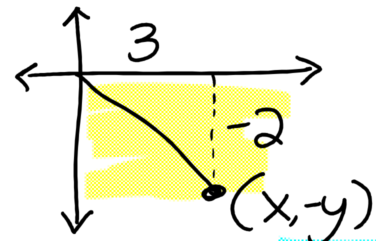
$\tan\theta = \frac{y}{x} = \frac{-2}{3}$

$y = -2 \quad x = 3$

$r^2 = 3^2 + (-2)^2$

$= 9 + 4$

$r^2 = 13 \quad r = \sqrt{13}$



$\sin\theta = \frac{y}{r} = \frac{-2}{\sqrt{13}} = \frac{-2\sqrt{13}}{13}$

Complete Assignment Questions #4 - #11

#1-9

Assignment

1. The point (8, –6) lies on the terminal arm of an angle θ in standard position. Determine the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$.

2. The point (–1, –3) lies on the terminal arm of an angle θ in standard position. Determine the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$.

3. The point (–16, 63) lies on the terminal arm of an angle A in standard position. Determine the exact value of cos A.

4. If $\cos\theta = \frac{12}{13}$ and $270° \le \theta \le 360°$, then find the exact values of $\sin\theta$ and $\tan\theta$.

5. If $\sin\theta = -\frac{4}{7}$ and $\cos\theta$ is negative, then find the exact value of $\tan\theta$.

6. If $\tan A = -\frac{15}{}$ and $0° < A < 180°$, then find the values of $\sin A$ and $\cos A$.

6. If $\tan A = -\frac{15}{8}$ and $0° \leq A \leq 180°$, then find the values of $\sin A$ and $\cos A$.

7. If $\tan B = 0.8$ and $\cos B$ is negative, then find the exact value of $\sin B$.

8. If $\sin X = -\frac{1}{4}$ and $\tan X$ is negative, express $\cos X$ as an exact value.

9. Solve for the required ratios in each of the following. Express each answer as an exact value with a rational denominator.

a) Angle θ terminates in the second quadrant. If $\tan \theta = -\frac{\sqrt{3}}{5}$, find $\sin \theta$ and $\cos \theta$.

value with a rational denominator.

a) Angle θ terminates in the second quadrant. If $\tan\theta = -\frac{\sqrt{3}}{5}$, find $\sin\theta$ and $\cos\theta$.

b) Angle θ terminates in the fourth quadrant. If $\tan\theta = -\frac{\sqrt{3}}{5}$, find $\sin\theta$ and $\cos\theta$.

Multiple Choice

10. If $\cos A = -\frac{7}{25}$ and $180° \leq A \leq 270°$, then the values of $\sin A$ and $\tan A$ respectively are

A. $-\frac{24}{25}$ and $-\frac{24}{7}$

B. $-\frac{24}{25}$ and $\frac{24}{7}$

C. $-\frac{24}{25}$ and $\frac{7}{24}$

D. $\frac{24}{25}$ and $\frac{24}{7}$

11. Angle P has a terminal arm in the third quadrant. If $\tan P = \dfrac{1}{\sqrt{3}}$, the value of $\sin P - \cos P$ is

A. $\dfrac{1-\sqrt{3}}{2}$

B. $\dfrac{\sqrt{3}-1}{2}$

C. $\dfrac{1+\sqrt{3}}{2}$

D. $\dfrac{-1-\sqrt{3}}{2}$

Answer Key

1. $\sin\theta = -\dfrac{3}{5}$ $\cos\theta = \dfrac{4}{5}$ $\tan\theta = -\dfrac{3}{4}$

2. $\sin\theta = -\dfrac{3\sqrt{10}}{10}$ $\cos\theta = -\dfrac{\sqrt{10}}{10}$ $\tan\theta = 3$

3. $\cos A = -\dfrac{16}{65}$

4. $\sin\theta = -\dfrac{5}{13}$ $\tan\theta = -\dfrac{5}{12}$

5. $\tan\theta = \dfrac{4\sqrt{33}}{33}$

6. $\sin A = \dfrac{15}{17}$ $\cos A = -\dfrac{8}{17}$

7. $\sin B = -\dfrac{4\sqrt{41}}{41}$

8. $\cos X = \dfrac{\sqrt{15}}{4}$

6. $\sin A = \frac{15}{17}$ $\cos A = -\frac{8}{17}$ **7.** $\sin B = -\frac{4\sqrt{41}}{41}$ **8.** $\cos X = \frac{\sqrt{15}}{4}$

9. a) $\sin\theta = \frac{\sqrt{21}}{14}$ $\cos\theta = -\frac{5\sqrt{7}}{14}$ **b)** $\sin\theta = -\frac{\sqrt{21}}{14}$ $\cos\theta = \frac{5\sqrt{7}}{14}$

10. B **11.** B