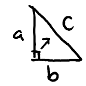
Trigonometry - Angles and Ratios Lesson #2:
Trigonometric Ratios for Angles from 0° to 360°

Pythagorean Theorem

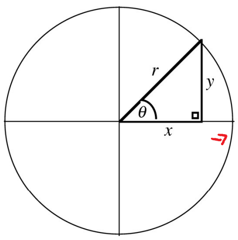


The traditional formula for the Pythagorean Theorem is $c^2 = a^2 + b^2$.

In trigonometry, we use x, y, and r instead of a, b, and c.

The point $P(x, y)$ lies on the terminal arm of angle θ.

The distance from the origin to point P is r, the radius of the circle formed by the rotation.



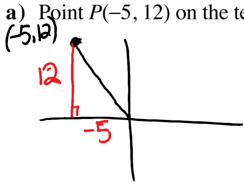
• (x, y)

$x^2 + y^2 = r^2$, where $r > 0$

$(0,0)$

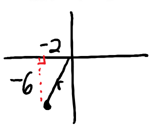
Sketch the rotation angle in standard position, and calculate the exact distance from the origin to the given point. Where appropriate, write the answer in simplest mixed radical form.

a) Point $P(-5, 12)$ on the terminal arm of angle θ.



$r^2 = x^2 + y^2$
$= (-5)^2 + 12^2$
$= 25 + 144$
$\sqrt{r^2} = \sqrt{169}$
$r = \sqrt{169} = 13$

b) Point $Q(-2, -6)$ on the terminal arm of angle A.



$(-2)^2 + (-6)^2 = r^2$
$4 + 36 = r^2$
$r^2 = 40$
$r = \sqrt{40} = \boxed{2\sqrt{10}}$

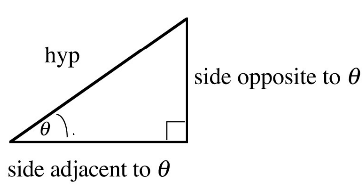
Trigonometric Ratios

Complete the following:

sine ratio ⇒ $\sin\theta = \frac{O}{H}$

cosine ratio ⇒ $\cos\theta = \frac{A}{H}$

tangent ratio ⇒ $\tan\theta = \frac{O}{A}$



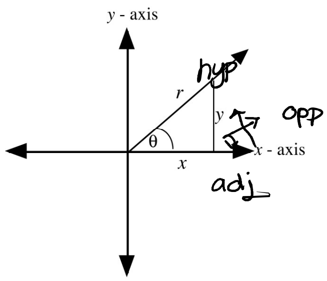
These ratios are called the ***Primary Trigonometric Ratios*** and can be remembered by the acronym **SOHCAHTOA**.

Write the primary trigonometric ratios for angle θ in terms of x, y, and r.

y - axis

y

Class Ex. #2 Write the primary trigonometric ratios for angle θ in terms of x, y, and r.



$\sin\theta = \frac{y}{r}$

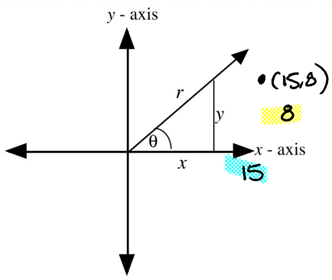
$\cos\theta = \frac{x}{r}$

$\tan\theta = \frac{y}{x}$

Note **You should memorize these formulas.**

Some students use a phrase like "**s**even **y**ellow **r**abbits" to remember **sin** $\theta = \frac{\boldsymbol{y}}{\boldsymbol{r}}$.

Class Ex. #3 The point (15, 8) lies on the terminal arm of an angle θ as shown. Calculate the value of r, and hence determine the exact values of the primary trigonometric ratios.



$r^2 = 15^2 + 8^2$

$r^2 = 289$

$r = 17$

$\sin\theta = \frac{8}{17}$

$\cos\theta = \frac{15}{17}$

$\tan\theta = \frac{8}{15}$

Complete Assignment Questions #1 - #5

$\tan^{-1}$ $\tan^{-1}$

$\tan\theta = \sqrt{3}$

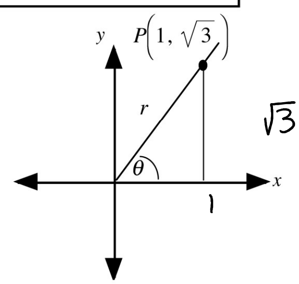
$\theta = \tan^{-1}\sqrt{3}$

$\theta = 60$

Investigating Trigonometric Ratios for Angles Between 90° and 360°

Part 1

Consider an angle θ in standard position with the point $P(1, \sqrt{3})$ on the terminal arm.



a) Show that the value of θ is 60°.

$x = 1$ $\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{1} = \sqrt{3}$

$y = \sqrt{3}$

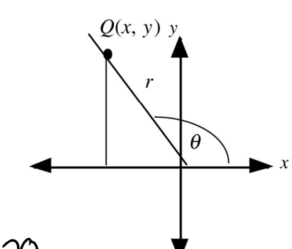
b) Calculate the value of r. $r^2 = 1^2 + \sqrt{3}^2 = 1 + 3$ $r^2 = 4$ $r = 2$

c) Complete the following, using $x =$ _____ , $y =$ _____ $\sqrt{3}$, and $r =$ _____ 2.

$\sin 60° = \frac{y}{r} = \frac{\sqrt{3}}{2}$ $\cos 60° = \frac{x}{r} = \frac{1}{2}$ $\tan 60° = \frac{y}{x} = \frac{\sqrt{3}}{1} = \sqrt{3}$

Part 2

The rotation angle in Part 1 is reflected in the y-axis.



Complete the following:

a) The point $Q(x, y)$ has coordinates $Q($, $-)$. $\sqrt{3}$ $r = 2$

b) The reference angle is _____ 60 and the rotation angle is _____ 120

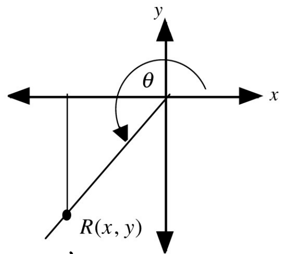
c) $\sin 120° = \frac{y}{} = \frac{\sqrt{3}}{2}$ $\cos 120° = \frac{x}{} = \frac{-1}{2}$ $\tan 120° = \frac{y}{} = \frac{\sqrt{3}}{1} = -\sqrt{3}$

b) The reference angle is ___60 and the rotation angle is ___120

c) $\sin 120° = \frac{y}{r} = \frac{\sqrt{3}}{2}$ $\quad \cos 120° = \frac{x}{r} = \frac{-1}{2}$ $\quad \tan 120° = \frac{y}{x} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$

d) Confirm these trigonometric ratios on your calculator.

Part 3



The rotation angle in Part 1 is reflected in both the x-axis and the y-axis.

Complete the following:

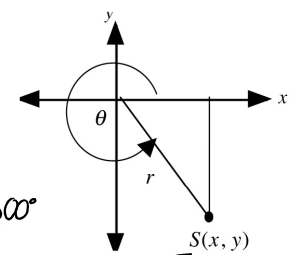
a) The point $R(x, y)$ has coordinates $R($, $-1)$. $-\sqrt{3}$ $\quad r = 2$

b) The reference angle is ___60° and the rotation angle is ___240°.

c) $\sin 240° = \frac{y}{r} = \frac{-\sqrt{3}}{2}$ $\quad \cos 240° = \frac{x}{r} = \frac{-1}{2}$ $\quad \tan 240° = \frac{y}{x} = \frac{-\sqrt{3}}{-1} = \sqrt{3}$

d) Confirm these trigonometric ratios on your calculator.

Part 4



The rotation angle in Part 1 is reflected in the x-axis.
Complete the following:

a) The point $S(x, y)$ has coordinates $S($, $1)$. $\sqrt{3}$ $\quad r = 2$

b) The reference angle is ___60 and the rotation angle is ___300°

c) $\sin 300° = \frac{y}{r} = \frac{-\sqrt{3}}{2}$ $\quad \cos 300° = \frac{x}{r} = \frac{1}{2}$ $\quad \tan 300° = \frac{y}{x} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$

d) Confirm these trigonometric ratios on your calculator.

Observations

- The trigonometric ratios for angles between 90° and 360° are either the trigonometric ratios of the reference angle, or the negative of the trigonometric ratios of the reference angle.
- The sign of the trigonometric ratios depends on the quadrant and whether x and y are positive or negative.

Determining the Sign of a Trigonometric Ratio

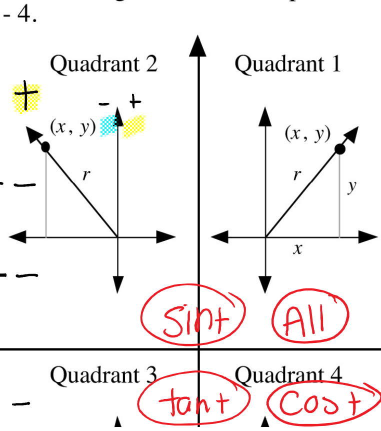
a) In quadrant 1, draw the rotation angle θ in standard position and complete the table.
b) Repeat for quadrants 2 - 4.

Quadrant 2

$\sin\theta =$	$\frac{y}{r}$	$\frac{+}{+}$	+
$\cos\theta =$	$\frac{x}{r}$	$\frac{-}{+}$	−
$\tan\theta =$	$\frac{y}{x}$	$\frac{+}{-}$	−

Quadrant 1

$\sin\theta = \frac{y}{r}$	$\frac{+}{+}$	+	+
$\cos\theta =$	$\frac{x}{r}$	$\frac{+}{+}$	+
$\tan\theta =$	$\frac{y}{x}$	$\frac{+}{+}$	+



Quadrant 3

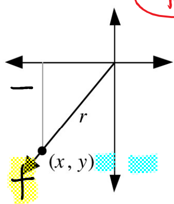
$\sin\theta =$	$\frac{y}{r}$	$\frac{-}{+}$	−

Quadrant 4

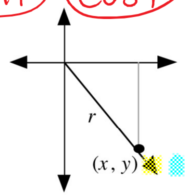
$\sin\theta =$	$\frac{y}{r}$	$\frac{-}{+}$	−

$\sin\theta =$	$\frac{y}{r}$	$\frac{-}{+}$	−
$\cos\theta =$	$\frac{x}{r}$	$\frac{-}{+}$	−
$\tan\theta =$	$\frac{y}{x}$	$\frac{-}{-}$	+

Quadrant 3 tan+



Quadrant 4 cos+



$\sin\theta =$	$\frac{y}{r}$	$\frac{-}{+}$	−
$\cos\theta =$	$\frac{x}{r}$	$\frac{+}{+}$	+
$\tan\theta =$	$\frac{y}{x}$	$\frac{-}{+}$	−

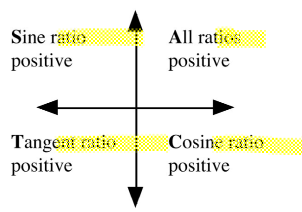
c) Complete the following statements using the results from a) and b).

i) Sine ratios have **positive** values in quadrants 1 and 2.
ii) Cosine ratios have **positive** values in quadrants 1 and 4.
iii) Tangent ratios have **positive** values in quadrants 1 and 3.
iv) Sine ratios have **negative** values in quadrants 3 and 4.
v) Cosine ratios have **negative** values in quadrants 2 and 3.
vi) Tangent ratios have **negative** values in quadrants 2 and 4.

CAST Rule

The results can be memorized by:

- the **CAST** rule or
- by remembering to "**A**dd **S**ugar **T**o **C**offee"



Class Ex. #4

Determine, without using technology, whether the given trigonometric ratios are positive or negative.

a) sin 340° Q4 negative
b) tan 227° S A T C Q3 positive
c) sin 88° positive
d) cos 235° S A T C negative
e) cos 308° positive
f) tan 123° negative

Trigonometric Ratios of an Angle in Terms of the Reference Angle

The trigonometric ratios for any angle are either the trigonometric ratios of the reference angle, or the negative of the trigonometric ratios of the reference angle.

Use the following procedure:

i) Determine the sign of the ratio (positive or negative).
ii) Determine the measure of the reference angle.
iii) Combine i) and ii).

To write cos 260° as the cosine of an acute angle using the above procedure, we have

i) negative ii) 80° (260−180) iii) cos 260° = – cos 80°.

The result can be verified on a calculator.

cos 260 = −0.17
cos 80 = 0.17

Class Ex. #5 Rewrite as the same trigonometric function of an acute angle.

Class Ex. #5

Rewrite as the same trigonometric function of an acute angle.

a) sin 140°

S A
T C
ref. angle 40°
sin ⊕
sin140 = $\boxed{\sin 40}$

b) tan 323°

360-323 = 37°
tan ⊖ in Q4
$\boxed{-\tan 37°}$

c) cos 165°

180-165 = 15
cos ⊖
$\boxed{-\cos 15}$

d) sin 287°

360-287 = 73
sin ⊖ Q4
-sin73

e) cos 308°

S A
T C
360-308 = 52°
cos ⊕
$\boxed{\cos 52}$

f) tan 199°

199-180 = 19
tan ⊕
$\boxed{\tan 19}$

Patterns in Trigonometric Ratios

We have the following pattern of results relating the trigonometric ratios of rotation angles to the trigonometric ratios of reference angles.

Let $x°$ be the reference angle for an angle in standard position.

$\sin(180 - x)° = \sin x°$ $\quad\cos(180 - x)° = -\cos x°$ $\quad\tan(180 - x)° = -\tan x°$

$\sin(180 + x)° = -\sin x°$ $\quad\cos(180 + x)° = -\cos x°$ $\quad\tan(180 + x)° = \tan x°$

$\sin(360 - x)° = -\sin x°$ $\quad\cos(360 - x)° = \cos x°$ $\quad\tan(360 - x)° = -\tan x°$

Complete Assignment Questions #6 - #11 and the Group Investigation.

Assignment

#1-8

1. Sketch the rotation angle in standard position, and calculate the exact distance from the origin to the given point.

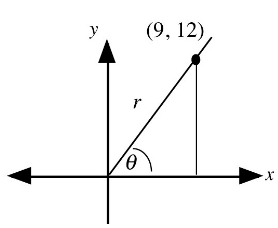
a) Point $P(15, -8)$ on the terminal arm of angle θ.

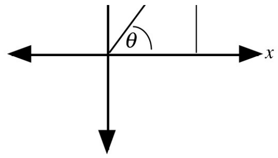
b) Point $Q(-24, -7)$ on the terminal arm of angle B.

2. Point $P(x, y)$ is on the terminal arm of angle θ in standard position. The distance $OP = r$, where O is the origin. Express the three primary trigonometric ratios in terms of x, y, and r.

$\sin\theta =$ $\qquad\cos\theta =$ $\qquad\tan\theta =$

3. The point (9, 12) lies on the terminal arm of an angle θ as shown. Calculate the value of r, and hence determine the exact values of the primary trigonometric ratios.





4. The point (5, 4) lies on the terminal arm of an angle θ. Determine the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$. Answer as an exact radical with a rational denominator.

5. The point (6, 12) lies on the terminal arm of an angle θ. Determine the exact values of $\sin\theta$, $\cos\theta$, and $\tan\theta$. Answer as a mixed radical in simplest form with a rational denominator.

6. In which quadrant(s) does the terminal arm of θ lie if:

a) $\sin\theta$ is positive? **b)** $\tan\theta$ is positive? **c)** $\cos\theta$ is negative?

d) both $\sin\theta$ and $\tan\theta$ are negative? **e)** $\cos\theta$ is positive and $\sin\theta$ is negative?

7. Determine, without using technology, whether the given trigonometric ratios are positive or negative.

a) cos 310° **b)** sin 94° **c)** tan 265°

d) sin 288° **e)** tan 109° **f)** cos 207°

8. Rewrite as the same trigonometric function of a positive acute angle.

a) sin 205° = **b)** tan 193° =

c) cos 97° = **d)** sin 156° =

e) cos 321° = **f)** tan 340° =

e) $\cos 321° =$

f) $\tan 340° =$

Multiple Choice

9. Without using technology, determine which of the following has a different sign from the others.

A. $\tan 255°$
B. $\sin 272°$
C. $\cos 175°$
D. $-\tan 75°$

10. Without using technology, determine which of the following has the same value as $\cos 297°$.

A. $\cos 27°$
B. $\cos 117°$
C. $-\cos 243°$
D. $-\cos 63°$

Numerical Response

11. Consider angles A, B, and C such that $\cos A = \cos 217°$, $\tan B = \tan 298°$, and $\sin C = \sin 7°$, where $0° \leq A \leq 360°$, $0° \leq B \leq 360°$, and $0° \leq C \leq 360°$.

The value of $A + B + C$ is _____.

(Record your answer in the numerical response box from left to right.)

Group Investigation

The following problems are a lead in to the next lesson.

a) Sketch an angle of 30° in standard position with the point $P(\sqrt{3}, 1)$ on the terminal arm.

Without using technology, explain and carry out a strategy to determine the exact trigonometric ratios of three different angles greater than 90° and less than 360°.

b) Consider an angle A in standard position with $\sin A = -\frac{3}{5}$ and $0° \leq A \leq 360°$.

Without using technology, explain and carry out a strategy to determine the exact values of $\cos A$ and $\tan A$.

Answer Key

1. a) 17 **b)** 25

2. $\sin\theta = \frac{y}{r}$ $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{y}{x}$

3. $r = 15$, $\sin\theta = \frac{4}{5}$ $\cos\theta = \frac{3}{5}$ $\tan\theta = \frac{4}{3}$

4. $\sin\theta = \frac{4\sqrt{41}}{41}$ $\cos\theta = \frac{5\sqrt{41}}{41}$ $\tan\theta = \frac{4}{5}$

5. $\sin\theta = \frac{2\sqrt{5}}{5}$ $\cos\theta = \frac{\sqrt{5}}{5}$ $\tan\theta = 2$

6. a) 1 or 2 **b)** 1 or 3 **c)** 2 or 3 **d)** 4 **e)** 4

7. a) Positive **b)** Positive **c)** Positive **d)** Negative **e)** Negative **f)** Negative

8. a) –sin 25° **b)** tan 13° **c)** –cos 83° **d)** sin 24° **e)** cos 39° **f)** –tan 20°

9. A **10.** C **11.**

4	3	4	

Group Investigation

a) $\sin 150° = \frac{1}{2}$ $\cos 150° = -\frac{\sqrt{3}}{2}$ $\tan 150° = -\frac{\sqrt{3}}{3}$

$\sin 210° = -\frac{1}{2}$ $\cos 210° = -\frac{\sqrt{3}}{2}$ $\tan 210° = \frac{\sqrt{3}}{3}$

$\sin 330° = -\frac{1}{2}$ $\cos 330° = \frac{\sqrt{3}}{2}$ $\tan 330° = -\frac{\sqrt{3}}{3}$

b) In quadrant three, $\cos A = -\frac{4}{5}$ and $\tan A = \frac{3}{4}$.

In quadrant four, $\cos A = \frac{4}{5}$ and $\tan A = -\frac{3}{4}$.