Logical Reasoning and Set Theory Lesson #1:
Connecting Words "And", "Or", "Not"

Overview

In this unit, we will explore the subject of set theory, the mathematical study of collections of objects. We will be introduced to the connecting words "and", "or", and "not", and their counterparts in set theory. We will also work with Venn diagrams and use them to illustrate concepts in set theory, including the empty set, disjoint sets, subsets, and universal sets.

Statements

In mathematics, we deal with statements. A **statement** is a sentence that has a truth value: it is either true or false. A sentence that may be judged true by one person and false by another is not considered a statement - it is an opinion.

Consider the following:

- The city of Vancouver is in the province of British Columbia.
  (This statement is true - it is a fact.)

- The city of Vancouver is on the planet Mars.
  (This statement is false.)

- Vancouver is the best place in the world to live.
  (This is not a statement - it is an opinion.)

- How do you get to Vancouver?
  (This is not a statement - it is a question.)

Class Ex. #1   Determine whether each of the following is a statement.
For each statement, list its truth value.

a) The number 4 is a prime number.

Statement false

b) Driving in a car is the best mode of transportation.

not a statement

c) If you double an odd integer, then you get an even integer.

$3 = 6 \quad 5 = 10 \quad 7 = 14 \quad 2n$

statement true

The Negation of a Statement Using the Word "Not"

The **negation of a statement** is the exact opposite of the statement.

The word "not" is often used to form the negation of a statement.
For instance, one way to write the negation of the statement

> "The city of Vancouver is in the province of Alberta"   F

is to write

> "The city of Vancouver is not in the province of Alberta".   T

Note that the original statement is <u>false</u>, while the negation is <u>true</u>.

- In every case where the original statement is true, the negation will be false.
- In every case where the original statement is false, the negation will be true.

Note It would not be correct to say that the negation of the above statement is a statement like "The city of Vancouver is in the province of Saskatchewan." Both this statement and the original statement are false. **A negation must always have the opposite truth value to the original statement.**

Class Ex. #2 A student made the following false statement: "All triangles are isosceles."
Which of the statements below is the negation of the above statement?

   A. All triangles are not isosceles.
   (B.) Not all triangles are isosceles.
   C. All triangles are scalene or equilateral.

Class Ex. #3 Write the negations of the following statements.

a) It is 20°C outside.   **b)** It is not Saturday today.

It is not 20°C outside.   It is Saturday today.

Compound Statements

A **compound statement** is a statement formed by combining two or more statements.

The words "and" or "or" are often used to form compound statements.

For example,

> "It is 20°C outside and it is not Saturday today."
> "It is 20°C outside or it is not Saturday today."

are compound statements.

Forming a Compound Statement Using the Word "and"

The **conjunction** of two statements can be formed by using the word "and".
A conjunction is true only if both of the original statements are true, and it is false otherwise.

- The conjunction of "blue is a colour" with "ice is a solid" is the statement "blue is a colour and ice is a solid". It is true because both original statements are true.

- The conjunction of "wet is a colour" with "ice is a solid" is the statement "wet is a colour and ice is a solid". It is false because one of the original statements is false. ✗

- The conjunction of "wet is a colour" with "ice is a liquid" is the statement "wet is a colour and ice is a liquid". It is false because at least one of the original statements is false. In fact, both are false. ✗

Forming a Compound Statement Using the Word "or"

In everyday English, we use the word "or" in two different ways.
The following examples illustrate this important difference.

Use of "or" in Everyday English

i) "In order for Drew to have the necessary prerequisites to go to college next term, he needs a pass in Grade 12 Physics or a pass in Grade 12 Chemistry this semester." This statement means that Drew will have the necessary prerequisites if;
- he passes Grade 12 Physics alone, or
- he passes Grade 12 Chemistry alone, or
- he passes both Grade 12 Physics and Grade 12 Chemistry. → *and/or*

This is an example of the (inclusive) use of "or" in everyday language.

ii) John asked Helen how she was getting to school tomorrow. She replied, "I will take the bus or drive myself."
This means that Helen either takes the bus or she drives, but not both.

This is an example of the (exclusive) use of "or" in everyday language.

And: both true
or: only one
has to be true

Use of "or" in Mathematics

In mathematics, including set theory, when we use the word "or" , we always mean the inclusive version of "or".

The **disjunction** of two statements is formed by placing the word "or" between them.

A disjunction is only false if both statements are false. A disjunction is true if any of the following happens:
- the first statement is true,
- the second statement is true, or,
- both statements are true.

Consider the following statement:

"Ivan rolled a pair of dice and the sum of the two numbers was an odd number or a prime number."

This means that the sum was
- an odd number, or
- a prime number, or
- both an odd number and a prime number.

This is an example of the (inclusive) use of "or" in mathematics.

Class Ex. #4

Consider the natural numbers x that are less than or equal to 15. **1-15**
List the numbers that satisfy each of the following conditions:

a) x is odd.

1, 3, 5, 7, 9, 11, 13, 15

b) x is prime.

2, 3, 5, 7, 11, 13

c) x is odd and x is prime.

3, 5, 7, 11, 13

d) x is odd or x is prime.

1, 2, 3, 5, 7, 9, 11, 13, 15

e) x is a factor of 3 or 5 or 7.

1, 3, 5, 7

f) x is a factor of 3 and 5 and 7.

1

g) x is a multiple of 3 or 5 or 7.

3, 6, 9, 12, 15, 5, 10, 7, 14

h) x is a multiple of 3 and 5 and 7.

Complete Assignment Questions #1 - #9

#1 -7

Assignment

1. Determine which of the following sentences are statements.
    Classify each statement as true or false.

    a) Cherry pie is delicious.

    b) Cherry pie is a food.

    c) Is Grade 12 math the best subject in high school?

    d) Jupiter is the closest planet to the sun.

    e) A pentagon has seven sides.

    f) Do quadratic functions have degree 2?

    g) The weather is too cold in Edmonton.

    h) There are 100 centimetres in a metre.

2. Consider the statements in question #1.

    a) Write the negation of each statement.

    b) Assess the truth value of each negation. What do you notice?

3. **a)** Circle the correct alternative:

- The conjunction of two statements is a compound statement formed by using "and"/ "or".
- The disjunction of two statements is a compound statement formed by using "and"/ "or".

    b) Take the statements in question #1b) and #1d) and form their conjunction.
       Is the resulting statement true or false?

    c) Take the statements in question #1e) and #1h) and form their disjunction.
       Is the resulting statement true or false?

4. **a)** List the natural numbers less than 20 that are divisible by 4.

    b) List the natural numbers less than 20 that are divisible by 6.

    c) List the natural numbers less than 20 that are divisible by 4 and 6.

    d) List the natural numbers less than 20 that are divisible by 4 or 6.

5. **a)** Give an example of the <u>inclusive</u> use of "or" in a compound statement.

    b) Give an example of the <u>exclusive</u> use of "or" in a compound statement.

6. Consider the following statements.
   In each case, state all the values of the variable that make the statement true.
   Each variable is restricted to natural numbers less than or equal to 15.

   a) a is a prime number and a is even.

   b) b is a factor of 12 and b is a multiple of 3.

   c) c is a factor of 12 and c is not a multiple of 3.

   d) d is a factor of 12 or d is a multiple of 3.

   e) e is not a prime number.

   f) f is an odd number, and f is a factor of 5 or a factor of 9.

   g) g is an odd number, and it is not the case that g is a factor of 5 or a factor of 9.

   h) h is an even number and h is a factor of 5.

7. Given statement A and statement B, complete the following table
   by filling either True or False in each box.

A	B	A and B	A or B	not A
True	True			
True	False			
False	True			
False	False			

Use the following information to answer the next two questions.

> Jeremy was given the following statements to consider.
> **Statement 1:** Ottawa is the capital of Canada.
> **Statement 2:** Canada is on the planet Venus.
> **Statement 3:** The city of Edmonton is in the province of Alberta.
> **Statement 4:** All scientists study biology or chemistry.

Multiple Choice **8.** Which of the following pairs of statements are true using the connecting word "and"?

- **A.** Statement 1 and Statement 2
- **B.** Statement 1 and Statement 3
- **C.** Statement 3 and Statement 4
- **D.** Statement 2 and Statement 4

9. Which of the following is an incorrect negation of Statement 4?

- **A.** Not all scientists study biology or chemistry.
- **B.** Some scientist does not study either biology or chemistry.
- **C.** All scientists study physics.
- **D.** There are scientists who do not study biology and do not study chemistry.

Answer Key

1 . b), d), e), and h) are statements. b) and h) are true, and d) and e) are false.

2 . Negation of b): Cherry pie is not a food.
Negation of d): Jupiter is not the closest planet to the sun.
Negation of e): A pentagon does not have seven sides.
Negation of h): There are not 100 centimetres in a metre.
Truth values: the negations of b) and h) are false and the negations of d) and e) are true. These are the opposite truth values to the original statements.

3 . **a)** • conjunction → and   • disjunction → or
b) Cherry pie is a food and Jupiter is the closest planet to the sun. This is false.
c) A pentagon has seven sides or there are 100 centimetres in a metre. This is true.

4 . **a)** 4, 8, 12, 16   **b)** 6, 12, 18   **c)** 12   **d)** 4, 6, 8, 12, 16, 18

5 . Answers may vary.
a) To become a citizen of Canada, you must speak English or French.
b) For tonight's concert, Angela has to wear black pants or a black dress.

6 . **a)** 2   **b)** 3, 6, 12   **c)** 1, 2, 4   **d)** 1, 2, 3, 4, 6, 9, 12, 15
e) 1, 4, 6, 8, 9, 10, 12, 14, 15   **f)** 1, 3, 5, 9   **g)** 7, 11, 13, 15   **h)** no solution

7 . See the table below.

A	B	A and B	A or B	not A
True	True	True	True	False
True	False	False	True	False
False	True	False	True	True
False	False	False	False	True

8 . B   **9 .** C