Lesson 4: Permutations with Repetitions

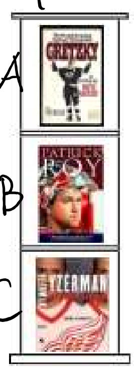
Permutations and Combinations Lesson #4:
Permutations with Repetitions

Investigating Permutations with Repetitions

Part 1

A book store is advertising signed copies of books about three famous Canadian hockey players in the Hockey Hall of Fame - Wayne Gretzky, Patrick Roy, and Steve Yzerman.

a) The books are arranged on a vertical display stand. One possible arrangement is shown. Sketch all possible arrangements of the three books. How many different arrangements are there?



1	2	3	4	5	6
A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

$3! = 6$

b) The bookstore sells out of the autographed Wayne Gretzky book including the display copy. The sales clerk replaces the Wayne Gretzky book with a second copy of the Patrick Roy book. One possible arrangement is shown. Sketch all possible arrangements of the three books. How many different arrangements are there?

B	B	C
B	C	B
C	B	B

3 arrangements

c) Complete the following statement:

The number of arrangements in b) is equal to the number of arrangements in a) divided by <u>2</u>.

Part 2

In lesson 2, we investigated the number of arrangements of all of the letters in a particular word. In every case, the letters in the word were different. In this part, we investigate what happens when there are letters which repeat within the same word.

To examine this scenario, consider the following four letter permutations of a word without repetitive letters, ROSE. Notice there are ${}_4P_4$, or 24 different arrangements.

$4! = 24$

ROSE	REOS	OSRE	SROE	SERO	EORS
ROES	RESO	OSER	SREO	SEOR	EOSR
RSOE	ORSE	OERS	SORE	EROS	ESRO
RSEO	ORES	OESR	SOER	ERSO	ESOR

a) Now, if we change the E in ROSE to an S, we get ROSS, a word with two letters which are repeating. If we change all the E's in the above list to S's, we will get all the arrangements for ROSS as shown in the list below.

ROSS	RSOS	OSRS	SROS	SSRO	SORS
ROSS	RSSO	OSSR	SRSO	SSOR	SOSR
RSOS	ORSS	OSRS	SORS	SROS	SSRO
RSSO	ORSS	OSSR	SOSR	SRSO	SSOR

$\frac{24}{2} = 12$

i) Complete the last column.

ii) There are no longer 24 different arrangements. Arrangements like ROSE and ROES from the first list both become ROSS in the second list and count as only **one arrangement**. The number of different arrangements of ROSS is 12.

iii) Notice that there are $\frac{1}{2}$ or $\frac{1}{2!}$ as many permutations of ROSS as there are of ROSE.

Hence, the number of permutations of ROSS is $\frac{4!}{2!}$, or 12.

b) If we change the O and E in ROSE to S, we get RSSS, a "word" with three repeating letters, with the arrangements shown below.

RSSS	RSSS	SSRS	SRSS	SSRS	SSRS
RSSS	RSSS	SSSR	SRSS	SSSR	SSSR
RSSS	SRSS	SSRS	SSRS	SRSS	SSRS
RSSS	SRSS	SSSR	SSSR	SRSS	SSSR

i) Complete the last column.

ii) Arrangements like ROSE, ROES, RSOE, RSEO, REOS, and RESO from the first list all become one arrangement of RSSS. The 24 original arrangements of ROSE is now reduced to 4 arrangements of RSSS.

iii) Notice that there are $\frac{1}{6}$ or $\frac{1}{3!}$ as many permutations of RSSS as there are of ROSE.

Hence, the number of permutations of RSSS is $\frac{4!}{3!}$, or 4.

There is a pattern in the lists on the previous page. If a letter appears twice in a word, we divide the total number of arrangements by 2!. If a letter appears three times in a word, we divide the total number of arrangements by 3!. The same pattern appeared in Part 1 of the investigation.

The following formula gives the number of permutations when there are **repetitions.**

> The number of permutations of n objects, where a are the same of one type, b are the same of another type, and c are the same of yet another type, can be represented by the expression below.
>
> $$\frac{n!}{a!\,b!\,c!}$$

Find the number of permutations of the letters of the word:

a) VANCOUVER

2V

$$\frac{9!}{2!} = 181\,440$$

b) MATHEMATICAL

2M 3A 2T

$$\frac{12!}{2!\,3!\,2!} = 19,958,400$$

$12! \div (2!\;3!\;2!)$

Brett bought a carton containing 10 mini boxes of cereal. There are 3 boxes of Corn Flakes, 2 boxes of Rice Krispies, 1 box of Coco Pops, 1 box of Shreddies, and the remainder are Raisin Bran. Over a ten day period, Brett plans to eat the contents of one box of cereal each morning.
How many different orders are possible if on the first morning he has Raisin Bran?

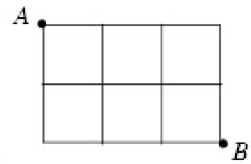
9 boxes

3 CF
2 RK
1 CP
1 S
2 RB

$$\frac{9!}{3!\,2!\,2!} = 15,120$$ different orders

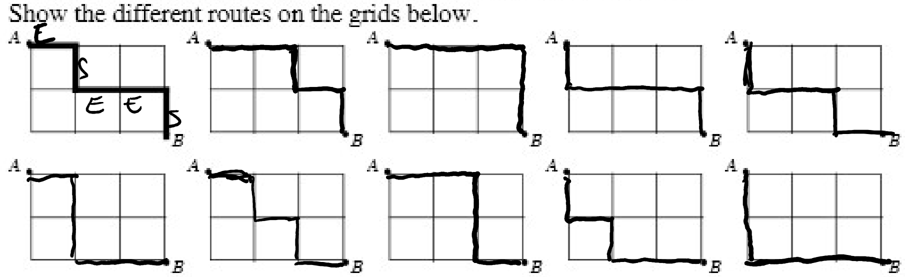
Complete Assignment Questions #1 - #5

Exploring Routes using Permutations and Patterns



Paul and Katherine are considering the following problem.
"Find the number of routes from A to B if routes must always move closer to B".

a) Paul arrived at the correct answer by tracing 10 routes on the grid. Show the different routes on the grids below.

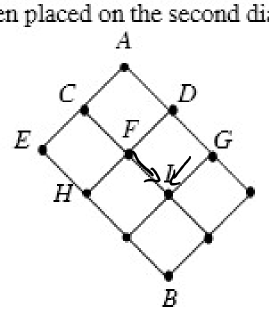


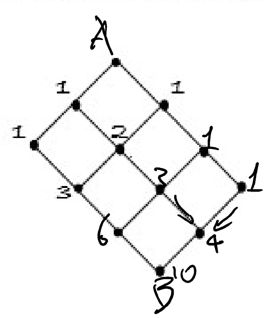
b) Katherine developed a quicker method. She realized that travel from A to B involves moving three units east, and two units south, and that the answer to the problem is the number of ways in which this can be done (i.e. EEESS in any order).
Note that Paul's first grid shows the route ESEES.

Solve the problem using Katherine's permutations with repetitions approach.

$$\frac{5!}{3!\,2!} = 10$$

c) An alternative approach is started below. The diagram has been rotated. There is one route from A to C, A to D, and A to E. There are two routes from A to F (ACF and ADF). There are three routes from A to H ($ACEH$, $ACFH$, and $ADFH$). All of these numbers have been placed on the second diagram.





b) How do the following answers relate?

i) F compared to C and D **ii)** H compared to E and F **iii)** I compared to F and G

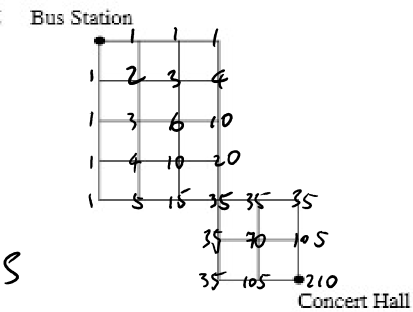
$F = C + D$ $H = E + F$ $I = F + G$

c) Use the relationship in b) to complete the second diagram and compare your answer with Paul and Katherine's answers.

The answer is also 10

A supervisor of the city bus department is determining how many routes there are from the bus station to the concert hall.

Determine the number of routes possible if the bus must always move closer to the concert hall.



a) using patterns

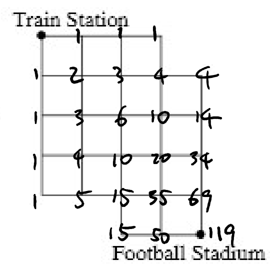
210

b) using permutations with repetitions

3E 4S 2E 2S

$$\frac{7!}{3!\,4!} \times \frac{4!}{2!\,2!} = 210$$

A taxi company is trying to find the quickest route during rush hour traffic from the train station to the football stadium.



a) Use patterns to determine the number of different routes that must be considered if at each intersection the taxi must always move closer to the football stadium.

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b) Explain why the permutations with repetitions formula $\frac{9!}{4!5!}$ cannot be used to determine the answer to this problem.

There are gaps in the rectangle that prevents some arrangements of 4E and 5S from being possible
e.g. EEEESSSSS is not possible

Complete Assignment Questions #6 - #12

Assignment

1, 3, 4, 5, 6a, 7, 8

Use the following information to answer question #1.

A **teapoy** is a small three-legged table or stand.

A **teapot** is a pot with a handle, spout, and lid, in which tea is brewed and from which it is poured.

A **teepee** is a portable conical tent made of skins, cloth, or canvas on a frame of poles.

1. **a)** Determine the number of permutations of all of the letters of the word **TEAPOY**.

b) Explain why the number of permutations of all of the letters of the word **TEAPOT** is not the same as the answer in a). Determine the number of permutations.

c) Determine the number of permutations of all of the letters of the word **TEEPEE**.

2. Explain why the number of different arrangements of the letters in the words **TOOTH** and **TEETH** is the same, but is different from the number of arrangements of the letters in the word **TENTH**.

3. How many different arrangements can be made using all of the letters of each word?

a) **COCHRANE** **b)** **WINNIPEG** **c)** **OSOYOOS**

4. A race at the Olympics has 8 runners. In how many orders can their countries finish if

a) there are 2 Canadian, 1 Russian, 1 German, 1 South African, and 3 American runners?

b) there are 1 Canadian, 2 British, 2 Ethiopian, 1 Algerian, and 2 Kenyan runners?

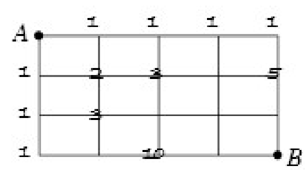
5. Naval signals are made by arranging coloured flags in a vertical line and the flags are then read from top to bottom. How many signals using six flags can be made if you have

a) 3 red, 1 green, and 2 blue flags? **b)** 2 red, 2 green, and 2 blue flags?

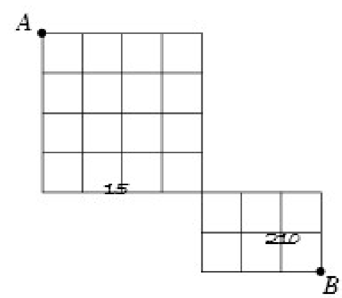
c) unlimited supplies of red, green, and blue flags?

6. a) Using patterns, determine the number of pathways from A to B if paths taken must always move closer to B. The numbers of pathways to certain points have been inserted as a guide.

i)



ii)

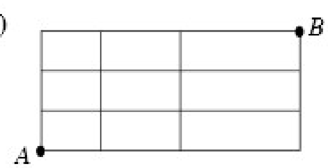


b) Verify the answers in a) using permutations with repetitions.

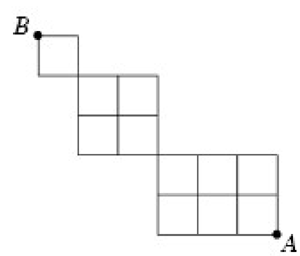
7. If routes must always move closer to B, determine the number of routes from A to B

i) using patterns **ii)** using permutations with repetitions

a)

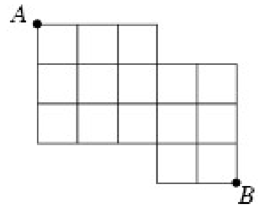


b)

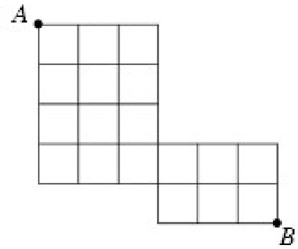


8. Determine the number of pathways from A to B if paths must always move closer to B.

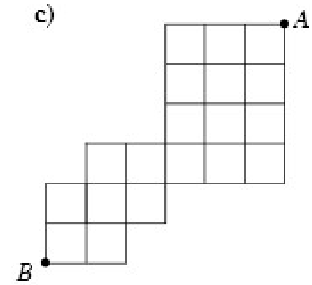
a)



b)



c)



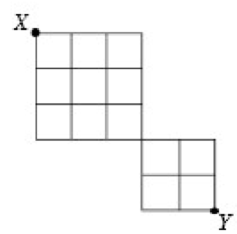
Multiple Choice

9. The number of different arrangements which can be made using all the letters of the word **SASKATOON** is

A. 720 **B.** 45 360

C. 362 880 **D.** 725 760

10. The number of pathways from X to Y if paths must always move closer to Y is



A. $\frac{6!}{3!\,3!} + \frac{4!}{2!\,2!}$ **B.** $\frac{6!}{3!\,3!} \times \frac{4!}{2!\,2!}$

C. $\frac{8!}{4!\,4!} + \frac{6!}{3!\,3!}$ **D.** $\frac{8!}{4!\,4!} \times \frac{6!}{3!\,3!}$

Answer Key

1. a) 720

b) In the 720 arrangements of the word TEAPOY in a) all the Y's are replaced by T's, resulting in duplicate arrangements. Therefore the number of arrangements is one half of 720 = 360.

c) 30

2. Both the words TOOTH and TEETH consist of five letters with two repeats of two letters. The number of arrangements is $\frac{5!}{2!2!} = 30$. The word TENTH has five letters, but only one repeat of two letters. The number of arrangements is $\frac{5!}{2!} = 60$.

3. a) 20 160 **b)** 10 080 **c)** 105 **4. a)** 3360 **b)** 5040

5. a) 60 **b)** 90 **c)** $3^6 = 729$ **6. i)** 35 **ii)** 700

7. a) 20 **b)** 720 **8. a)** 106 **b)** 260 **c)** 495 **9.** B **10.** B