Lesson 2: Factorial Notation and Permutations

Permutations and Combinations Lesson #2:
Factorial Notation and Permutations

Investigating Factorial Notation

Consider how many ways there are of arranging 6 different books side by side on a shelf. Using the fundamental counting principle, we calculate the product $6 \times 5 \times 4 \times 3 \times 2 \times 1$.

In mathematics this product can be written in a simplified form called factorial notation.

$6 \times 5 \times 4 \times 3 \times 2 \times 1$ is denoted by $6!$ ("6 factorial" or "factorial 6").

a) Without using a calculator, determine the value of:

i) $4! = 4\times3\times2\times1 = 24$ "4 factorial" ii) $3! \times 2!$ $3\times2\times1 \times 2\times1 = 12$

Notice that 6! can be written in a variety of ways.

$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$6! = 6 \times 5 \times 4 \times 3 \times 2!$

$6! = 6 \times 5 \times 4 \times 3!$

$6! = 6 \times 5 \times 4!$

$6! = 6 \times 5!$

The table above can be used to help us simplify quotients such as $\frac{6!}{4!}$.

$\frac{6!}{4!} = \frac{6 \times 5 \times \cancel{4!}}{\cancel{4!}} = 6 \times 5 = 30.$ $\frac{6\times5\times\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}}{\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}}$

b) Express 5! as a product in four different ways.

$5! = 5\times4!$ $5! = 5\times4\times3!$ $5! = 5\times4\times3\times2!$ $5! = 5\times4\times3\times2\times1$

c) Complete the following: **i)** $12! = 12 \times 11!$ **ii)** $8! = 8 \times 7 \times 6!$

d) Express as a quotient of factorials.

i) $6 \times 5 \times 4 = \frac{6!}{3!}$ **ii)** $15 \times 14 \times 13 \times 12 = \frac{15!}{11!} = \frac{15\times14\times13\times12\times\cancel{11!}}{\cancel{11!}}$

e) Without using a calculator, simplify

i) $\frac{10!}{7!} = \frac{10\times9\times8\times\cancel{7!}}{\cancel{7!}}$
$= 10\times9\times8 = 720$

ii) $\frac{8!}{5!3!} = \frac{8\times7\times6\times\cancel{5!}}{\cancel{5!}\,3!}$
$= \frac{8\times7\times6}{3\times2\times1} = \frac{8\times7\times\cancel{6}}{\cancel{6}} = 56$

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Factorial Notation

one smaller
two smaller

In general $n! = n(n-1)(n-2)(n-3) \ldots (3)(2)(1)$, **where $n \in W$.**

3 smaller

- Note that n has to be a **whole number**.
- $0! = 1$, and the reasoning for this will be given later in this lesson.

$n!$ can be written as a product in many different ways.

For example $n! = n(n-1)!$
$n! = n(n-1)(n-2)!$
$n! = n(n-1)(n-2)(n-3)!$, etc.

Complete the following.

a) $n! = \underline{\ n\ } \times (n-1)!$

b) $(n+2)! = \underline{(n+2)(n+1)}\,n!$

* always one less

c) $(n-4)(n-5)(n-6) \ldots (3)(2)(1) = \underline{(n-4)}!$

Simplify the following expressions. Leave the answer in product form where appropriate.

a) $\dfrac{(n+2)!}{n!}$

$$= \frac{(n+2)(n+1)\cancel{n!}}{\cancel{n!}}$$

$$= (n+2)(n+1)$$

b) $\dfrac{(n-2)!}{(n-1)!}$

$$\frac{\cancel{(n-2)!}}{(n-1)\cancel{(n-2)!}}$$

$$= \frac{1}{n-1}$$

c) $\dfrac{n!}{n(n-1)}$

$$= \frac{\cancel{n}\cancel{(n-1)}(n-2)!}{\cancel{n}\cancel{(n-1)}}$$

$$= (n-2)!$$

Algebraically solve the equation $\dfrac{n!}{(n-2)!} = 42$.

$$\boxed{\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}}} = 42$$

$$n(n-1) = 42$$

$$n^2 - n = 42$$

$$n^2 - n - 42 = 0$$

$$(n+6)(n-7) = 0$$

$$n = \cancel{-6}, 7$$

$$\boxed{n = 7}$$

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Factorial Notation on a Calculator

Many calculators have a factorial key which can be used to simplify calculations.

a) Use the factorial key on a calculator to evaluate the following.

$6! =$ 720 $\qquad 9! =$ 362 880

b) To simplify a quantity like $\frac{10!}{7!3!}$ on a calculator we can evaluate numerator and denominator separately and then divide, or enter it in one calculation using brackets.

i) Complete the work below.

$$\frac{10!}{7!3!} = \frac{3\ 628\ 800}{30\ 240} = \qquad \text{or} \qquad \frac{10!}{(7!3!)} =$$

ii) To verify without using a calculator, complete the work below.

$$\frac{10!}{7!3!} = \frac{10 \times 9 \times 8 \times 7!}{7! \times 6} =$$

Calculate the value of: **a)** $\frac{43!}{40!} = \frac{43 \cdot 42 \cdot 41 \cdot \cancel{40!}}{\cancel{40!}}$ **b)** $\frac{37!}{33!\,4!} = 66045$

$43! \div 40! = 74046$

Complete Assignment Questions #1 - #6

Permutations

An internet site's access code consists of three digits. Knowing the three digits is not enough to access the site. The digits have to be entered in the correct order. *The order of the arrangement of the digits is important.*

John cannot remember the access code, except that it contains the digits 3, 5 and 7.

- List all the arrangements of these three digits that John could use to determine the access code.

357, 375, 537, 573, 735, 753.
they are 6 permutations

There are six possible arrangements to consider and only one of them will access the site. This type of arrangement, where the order is important, is called a **permutation**.

A permutation is an **arrangement** of a set of elements in which **the order of the elements is important**.

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Permutations of "n" Different Elements Taken "n" at a Time

1! = 1
2! = 2
3! = 3·2·1 = 6
4! = 4·3·2·1 = 24

a) List all the arrangements of the letters of the word **CAT**.

CAT CTA ACT ATC TCA TAC

b) Write the number of permutations in factorial notation. 3! = 6

This is an example of the following general rule:

The number of permutations of "n" different elements taken all at a time is $n!$

Class Ex. #5

Determine the number of permutations of the letters of each word.

a) **REGINA**

6! = 720

b) **KELOWNA**

7! = 5040

Permutations of "n" Different Elements Taken "r" at a Time (r ≤ n)

a) Use the fundamental counting principle to determine how many three letter arrangements can be made from the letters of the word **GRAPHITE.**

8 7 6 = 336

In the example above we have found the number of permutations of 8 (n) elements taken 3 (r) at a time. This is denoted by ${}_8P_3$.

$$ {}_8P_3 \;=\; 8 \times 7 \times 6 \;=\; \frac{8 \times 7 \times 6 \times \cancel{5!}}{\cancel{5!}} \;=\; \frac{8!}{(8-3)!} $$

This is an example of the following general rule:

Permutate

The number of permutations of "n" different elements taken "r" at a time is

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

b) Use the ${}_nP_r$ key on a calculator to evaluate ${}_8P_3$. Verify using factorials.

n=8 r=3 "8 elements permutate to get 3"

$\boxed{8}\boxed{nPr}\boxed{3}\boxed{=}$ 336

$$ {}_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8\cdot7\cdot6\cdot\cancel{5!}}{\cancel{5!}} $$

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Defining 0!

If we replace r by n in the previous formula, we get the number of permutations of n elements taken n at a time. This we know is $n!$.

$${}_nP_n = n! = \frac{n!}{(n-n)!} = \frac{n!}{0!}$$

For this to be equal to $n!$ the value of $0!$ must be 1.

0! is defined to have a value of 1.

In a region, vehicle license plates consist of 2 different letters followed by 4 different digits. If the letters I, O, Y, and Z are not used, determine how many different license plates are possible by

a) the fundamental counting principle **b)** permutations

$\underbrace{\underline{22}\ \underline{21}}_{\text{2 different letters, not I, O, Y, Z}}\ \underbrace{\underline{10}\ \underline{9}\ \underline{8}\ \underline{7}}_{\text{4 different digits}} = 2{,}328{,}480 = {}_{22}P_2 \times {}_{10}P_4$

1-4, 5bc, 6a, 7, 8, 10, 11

In many cases involving simple permutations, the fundamental counting principle can be used in place of the permutation formulas.

Complete Assignment Questions #7 - #14

Assignment

1. Without using a calculator, determine the value of

a) $5!$ **b)** $\frac{10!}{8!}$ **c)** $\frac{99!}{100!}$

2. Express as single factorials.

a) $6 \times 5 \times 4 \times 3 \times 2 \times 1$ **b)** $9 \times 8 \times 7 \times 6!$ **c)** $(n+2)(n+1)n(n-1)\ \ldots \times 3 \times 2 \times 1$

3. Express as a quotient of factorials.

a) $9 \times 8 \times 7 \times 6$ **b)** $20 \times 19 \times 18$ **c)** $(n+2)(n+1)n$

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4. Use a calculator to determine the exact value of the following:

a) $10!$ **b)** $\dfrac{8!}{4!}$ **c)** $\dfrac{15!}{10!\,5!}$ **d)** $\left(\dfrac{25!}{21!}\right)\left(\dfrac{7!}{11!}\right)$

5. Simplify the following expressions. Leave the answer in product form where appropriate.

a) $\dfrac{n!}{n}$ **b)** $\dfrac{(n-3)!}{(n-2)!}$ **c)** $\dfrac{(n+1)!}{(n-1)!}$ **d)** $\dfrac{(3n)!}{(3n-2)!}$

6. Solve the equation.

a) $\dfrac{(n+1)!}{n!} = 6$ **b)** $(n+1)! = 6(n-1)!$

c) $\dfrac{(n+2)!}{n!} = 12$ **d)** $\dfrac{(n+1)!}{(n-2)!} = 20(n-1)$

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7. Determine the number of arrangements that can be made using all of the letters in the word

a) DOG **b)** DUCK **c)** SANDWICH **d)** CANMORE

8. Consider the number of five-digit numbers that can be made from the digits 2, 3, 4, 7, and 9 if no digit can be repeated. Express your answer using

a) factorial notation **b)** ${}_nP_r$ notation **c)** the fundamental counting principle

9. a) Use the formula for ${}_nP_r$ to show that ${}_7P_0 = 1$.

b) Explain why n must be greater than or equal to r in the notation ${}_nP_r$.

10. In each case determine the number of arrangements of the given letters by

i) using the fundamental counting principle **ii)** writing in ${}_nP_r$ form and evaluating

a) two letters from the word **GOLDEN** **b)** three letters from the word **CHAPTERS**

c) four letters from the word **WEALTH** **d)** one letter from the word **VALUE**

11. How many numbers (up to a maximum of four digit numbers) can be made from the digits 2, 3, 4, and 5 if no digit can be repeated?

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Multiple Choice **12.** In a ten-team basketball league, each team plays every other team twice, once at home and once away. The number of games that are scheduled is

A. 45

B. 90

C. 100

D. 180

13. The value of ${}_nP_2$ is

A. $\dfrac{n}{n-2}$

B. $\dfrac{n!}{2!}$

C. $\dfrac{n}{2}$

D. $n(n-1)$

Numerical Response **14.** In a competition on the back of a cereal packet, seven desirable qualities for a kitchen (eg. spaciousness, versatility, etc.) must be put in order of importance. The number of different entries that must be completed in order to ensure a winning order is _____ .

(Record your answer in the numerical response box from left to right.)

Answer Key

1. a) 120 **b)** 90 **c)** $\dfrac{1}{100}$ **2. a)** 6! **b)** 9! **c)** $(n+2)!$

3. a) $\dfrac{9!}{5!}$ **b)** $\dfrac{20!}{17!}$ **c)** $\dfrac{(n+2)!}{(n-1)!}$ **4. a)** 3 628 800 **b)** 1680 **c)** 3003 **d)** $\dfrac{115}{3}$

5. a) $(n-1)!$ **b)** $\dfrac{1}{n-2}$ **c)** $n(n+1)$ **d)** $3n(3n-1)$

6. a) $n = 5$ **b)** $n = 2$ **c)** $n = 2$ **d)** $n = 4$

7. a) 6 **b)** 24 **c)** 40 320 **d)** 5040

8. a) 5! **b)** ${}_5P_5$ **c)** $5 \times 4 \times 3 \times 2 \times 1 = 120$

9. a) ${}_7P_0 = \dfrac{7!}{(7-0)!} = \dfrac{7!}{7!} = 1$

b) You cannot arrange more elements than the number of elements there are to begin with.

10. a) ${}_6P_2 = 30$ **b)** ${}_8P_3 = 336$ **c)** ${}_6P_4 = 360$ **d)** ${}_5P_1 = 5$

11. 64 **12.** B **13.** D **14.**

5	0	4	0

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