Lesson 4: Independent/Dependent Events

Probability Lesson #4:
Independent/Dependent Events and the Event "$A \cap B$"

Investigating Independent Events

One card is drawn from a deck of cards and is replaced. A second card is then drawn. Consider the following events.

A = {the first card is a heart} B = {the second card is a heart}

a) Determine $P(A)$. $\frac{13}{52} = \frac{1}{4}$ **b**) Determine $P(B)$. $\frac{1}{4}$

The probability of event B does NOT depend on whether or not event A occurred. Events A and B are called **independent events**.

Independent Events

Two events are **independent** if the knowledge that one event has occurred has no effect on the probability of the other event occurring.

Investigating Dependent Events

One card is drawn from a deck of cards and is not replaced. A second card is then drawn. Consider the following events.

A = {the first card is a heart} B = {the second card is a heart}

a) Determine $P(A)$. $= \frac{1}{4}$ **b**) Why is it not possible to determine $P(B)$?

It depends on whether or not a heart was already pulled

The probability of event B depends on whether or not event A occurred. Events A and B are called **dependent events**.

Dependent Events

Two events are **dependent** if the knowledge that one event has occurred changes the probability of the other event occurring.

Classify the following events as dependent or independent.

a) The experiment is rolling a die and tossing a coin.
The first event is rolling 2 on the die and the second event is tossing tails on the coin.

independent

b) The experiment is choosing two cards without replacement from a standard deck .
The first event is that the first card is a king and the second event is that the second card is a king.

dependent

c) The experiment is choosing two cards with replacement from a standard deck.
The first event is that the first card is a king and the second event is that the second card is a king.

independent

Conditional Probability

Consider the scenario from the previous page.

One card is drawn from a deck of cards and is not replaced. A second card is then drawn. Consider the following events.

A = {the first card is a heart} B = {the second card is a heart}

$P(A) = \frac{13}{52} = \frac{1}{4}$, but $P(B)$ depends on whether the event A occurred or did not occur.

- We denote by $P(B|A)$ the **conditional probability** of the event B, given that the event A has occurred. ← prob of B if A happened

  In this example, $P(B|A) = \frac{12}{51}$ ← one less heart ← one less card

- We denote by $P(B|A')$ the **conditional probability** of the event B, given that the event A has **NOT** occurred. ← prob of B if A didn+ happen

  In this example, $P(B|A') = \frac{13}{51}$ ← all the hearts remain ← one less card

Investigating Conditional Probability

A red die and a blue die are tossed.
The outcomes are shown in the array.

Consider the following two events:

Event A = {the sum of the two dice is 10}
Event B = {the number on each die is the same}

Blue Die

Red Die	**1**	**2**	**3**	**4**	**5**	**6**
1	{1, 1}	{1, 2}	{1, 3}	{1, 4}	{1, 5}	{1, 6}
2	{2, 1}	{2, 2}	{2, 3}	{2, 4}	{2, 5}	{2, 6}
3	{3, 1}	{3, 2}	{3, 3}	{3, 4}	{3, 5}	{3, 6}
4	{4, 1}	{4, 2}	{4, 3}	{4, 4}	{4, 5}	{4, 6}
5	{5, 1}	{5, 2}	{5, 3}	{5, 4}	{5, 5}	{5, 6}
6	{6, 1}	{6, 2}	{6, 3}	{6, 4}	{6, 5}	{6, 6}

a) List the outcomes for the following events as subsets of the sample space.

i) $A = \{(6,4)\ (5,5), (4,6)\}$

ii) $B = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$ **iii)** $A \cap B = \{(5,5)\}$

b) State the following probabilities:

i) $P(A) = \frac{3}{36} = \frac{1}{12}$ **ii)** $P(B) = \frac{6}{36} = \frac{1}{6}$ **iii)** $P(B|A) = \frac{1}{3}$ ✓

iv) $P(A|B) = \frac{1}{6}$ **v)** $P(A \cap B) = \frac{1}{36}$ **vi)** $P(B \cap A) = \frac{1}{36}$

c) Verify the following:

i) $P(A \cap B) = P(A) \times P(B|A)$

$\frac{1}{36} = \frac{1}{12} \times \frac{1}{3}$

$\frac{1}{36} = \frac{1}{36}$ ✓

ii) $P(B \cap A) = P(B) \times P(A|B)$

$\frac{1}{36} = \frac{1}{6} \times \frac{1}{6}$

$\frac{1}{36} = \frac{1}{36}$ ✓

Note that $P(A \cap B) = P(B \cap A)$, even though the calculations are different.

Multiplication Law for Dependent Events

If the events A, B are dependent, then

$$P(A \cap B) = P(A) \cdot P(B \mid A)$$

"probability of A and B equals the prob of A times the probability of A then B"


Class Ex. #2

Two cards are drawn without replacement from a standard deck of 52 cards. ← dependent
Determine the probability of the following events.

a) Both cards are red. 1st card 2nd card

$$\left(\frac{26}{52}\right)\left(\frac{25}{51}\right)$$

$$\left(\frac{1}{2}\right)\left(\frac{25}{51}\right) = \frac{25}{102}$$

b) Neither card is a club. 39 non clubs

$$\left(\frac{39}{52}\right)\left(\frac{38}{51}\right)$$

$$= \left(\frac{3}{4}\right)\left(\frac{38}{51}\right) = \frac{114}{204} = \frac{19}{34}$$

c) The first card is a king and the second card is a five.

$$\underset{K}{\left(\frac{4}{52}\right)}\underset{5}{\left(\frac{4}{51}\right)} = \frac{4}{663}$$

d) One of the cards is a king and the other is a five.

K5, 5K

$$\frac{4}{663} + \frac{4}{663} = \frac{8}{663}$$

Multiplication Law for Independent Events

If the events A, B, are independent, then the knowledge that event A has occurred has no effect on the probability of the event B occurring. This means that $P(B \mid A) = P(B)$.

Therefore, we can adapt the above law for independent events.

If the events A, B are **independent**, then

$$P(A \cap B) = P(A) \cdot P(B)$$


Class Ex. #3

Two cards are drawn with replacement from a standard deck of 52 cards. ← independent
Determine the probability of the following events.

a) Both cards are red.

$$\left(\frac{26}{52}\right)\left(\frac{26}{52}\right) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

b) The first card is a king and, the second card is a five.

$$\left(\frac{4}{52}\right)\left(\frac{4}{52}\right) = \left(\frac{1}{13}\right)\left(\frac{1}{13}\right) = \frac{1}{169}$$

c) One of the cards is a king and the other is a five.

$$\underset{K5}{\frac{1}{169}} + \underset{5K}{\frac{1}{169}} = \frac{2}{169}$$

Note Students often confuse the concept of **mutually exclusive** events with the concept of **independent** events. These terms do NOT mean the same thing.

- The concept of **mutually exclusive events** involves whether or not two events can occur at the same time.

- The concept of **independent events** involves whether or not the occurrence of one event has an effect on the probability of the other event occurring.

Class Ex. #4

If $P(A) = \frac{1}{3}$, $P(B) = \frac{2}{5}$, and $P(A \cup B) = \frac{3}{5}$, use appropriate formulas to investigate whether event A and event B are

a) mutually exclusive events

If $P(A) + P(B) \neq P(A \cup B)$

then not mutually exclusive

$$\frac{1}{3} + \frac{2}{5}$$

$$\frac{5}{15} + \frac{6}{15} = \frac{11}{15} \neq \frac{3}{5}$$

not mutually exclusive

b) independent events

if $P(A \cap B) = P(A) \cdot P(B)$

then independent

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$= \frac{1}{3} + \frac{2}{5} - \frac{3}{5}$$

$$= \frac{5}{15} + \frac{6}{15} - \frac{9}{15} = \frac{2}{15}$$

$$\left(\frac{1}{3}\right)\left(\frac{2}{5}\right) = \frac{2}{15} \longleftrightarrow$$

$\checkmark$ independent

Class Ex. #5

Let A and B be events with $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, and $P(A \cap B) = \frac{1}{4}$.

Determine :

a) $P(B|A)$

$$\frac{P(A \cap B)}{P(A)} = \frac{\cancel{P(A)} \cdot P(B|A)}{\cancel{P(A)}}$$

$$\boxed{P(B|A) = \frac{P(A \cap B)}{P(A)}}$$

$$= \frac{1}{4} \div \frac{1}{2} \quad \frac{2}{4}$$

$$\boxed{= \frac{1}{2}}$$

b) $P(A|B)$

$$P(B \cap A) = P(B) \cdot P(A|B)$$

$$\boxed{P(A|B) = \frac{P(B \cap A)}{P(B)}}$$

$$= \frac{\frac{1}{4}}{\frac{1}{3}}$$

$$= \frac{1}{4} \div \frac{1}{3} = \boxed{\frac{3}{4}}$$

c) $P(A \cup B)$

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$$

$$= \frac{6}{12} + \frac{4}{12} - \frac{3}{12}$$

$$\boxed{= \frac{7}{12}}$$

The probability that Sophia will pass Math this semester is 0.7 and the probability that she will pass English this semester is 0.9. If these events are independent, determine the probability (to the nearest hundredth) that she will pass

a) Math and English

$P(M\cap E) = P(M)\cdot P(E)$
$= 0.7\cdot 0.9$
$= 0.63$

b) Math or English

$P(M\cup E) = P(M) + P(E) - P(M\cap E)$
$= 0.7 + 0.9 - 0.63$
$= 0.97$

c) Math but not English

$P(M\cap E') = P(M)\cdot P(E')$
$= 0.7\cdot 0.1$
$= 0.07$

d) neither Math nor English

$P(M'\cap E') = 0.3\cdot 0.1$
$= 0.03$

A number is selected from the first ten natural numbers.
Consider the events *M*, "a prime number is chosen", and *N*, "an odd number is chosen".

a) List the sample space and the set of outcomes for the events *M*, *N*, *M* ∩ *N*, and *M* ∪ *N*.

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$M = \{2, 3, 5, 7\}$
$N = \{1, 3, 5, 7, 9\}$
$M\cap N = \{3, 5, 7\}$
$M\cup N = \{1, 2, 3, 5, 7, 9\}$

b) State the probabilities of the events *M*, *N*, *M* ∩ *N*, and *M* ∪ *N*.

$P(M) = \frac{4}{10} = \frac{2}{5}$
$P(N) = \frac{5}{10} = \frac{1}{2}$
$P(M\cap N) = \frac{3}{10}$
$P(M\cup N = \frac{6}{10} = \frac{3}{5}$

c) Determine *P*(a prime number is chosen | an odd number is chosen) i.e. *P*(*M* | *N*) using the appropriate formula.

$P(M|N) = \frac{3}{5}$

$P(M|N) = \frac{\overset{5}{P(N\cap M)}}{P(N)} = \frac{3/10}{1/2} = \frac{3}{10} \div \frac{1}{2} = \frac{6}{10}$

d) Aidan said that a quicker method for determining the answer in c) would be to look at the number of odd numbers in the sample space and count how many of these were prime. Do you agree? yes

$\frac{3}{5}$ are prime / odd #s

e) Determine *P*(an odd number is chosen | a prime number is chosen).

$P(N|M) = \frac{3}{4}$

$P(N|M) = \frac{P(M\cap N)}{P(M)}$

$= \frac{\frac{3}{10}}{\frac{2}{5}} = \frac{3}{10} \div \frac{2}{5} = \frac{15}{20} = \frac{3}{4}$

Complete Assignment Questions #1 - #14

Assignment

1. Classify the following events as dependent or independent.

a) The experiment is to consider the height and weight of students.
The first event is that the student is greater than 1.8 m tall and the second event is that the student weighs more than 70 kg.

b) The experiment is choosing two cards with replacement from a standard deck.
The first event is that the first card is a jack and the second event is that the second card is a queen.

c) The experiment is choosing two cards without replacement from a standard deck.
The first event is that the first card is a seven and the second event is that the second card is a seven.

d) The experiment is rolling a die and rolling the die again.
The first event is that the number on the first roll is a six and the second event is that the number on the second roll is a two.

2. Consider two events such that $P(A) = \frac{1}{4}$, $P(B) = \frac{2}{5}$, and $P(A \cap B) = \frac{1}{10}$.

a) Are A, B mutually exclusive events? **b**) Are A, B independent events?

3. Consider two events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, and $P(A \cup B) = \frac{3}{4}$.

a) Are A, B mutually exclusive events? **b**) Are A, B independent events?

4. Let A and B be events with $P(A) = 0.6$, $P(B) = 0.4$, and $P(A \text{ and } B) = 0.3$.
Determine :

a) $P(A|B)$

b) $P(B|A)$

c) $P(A \cup B)$

5. Let A and B be events with $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{8}$, and $P(A \text{ or } B) = \frac{3}{4}$.
Determine:

a) $P(A \text{ and } B)$

b) $P(A|B)$

c) $P(B|A)$

6. Caden rolls a red die and a blue die. Determine the probability that the red die shows a 1 and the blue die shows a 5 or a 6.

7. The probabilities that Emma will pass Grade 12 math and Grade 12 physics this semester are 0.85 and 0.75 respectively. If these events are independent, determine the probability (to four decimal places) that she will pass

a) both math and physics

b) math but not physics

c) physics but not math

d) neither math nor physics

8. Two cards are drawn **with replacement** from a standard deck of 52 cards. Determine the probability of the following events.

a) both cards are spades **b)** both cards are sevens **c)** neither card is red

d) the first card is a club and the second card is a diamond

e) one of the cards is red and the other is black

9. Two cards are drawn **without replacement** from a standard deck of 52 cards. Determine the probability of the following events.

a) both cards are spades **b)** both cards are sevens **c)** neither card is red

d) the first card is a club and the second card is a diamond

e) one of the cards is red and the other is black

10. In a city school, 60% of students have blue eyes, 55% have dark hair, and 20% have neither blue eyes nor dark hair.

a) Determine the probability that a randomly selected student will have blue eyes and dark hair.

b) State, with a reason, if these two characteristics are independent.

11. A standard die is rolled.

a) State the outcomes associated with the following events

i) an odd number is rolled **ii)** an even number is rolled **iii)** a prime number is rolled

b) Determine:

i) P(an odd number is rolled | a prime number is rolled)

ii) P(a prime number is rolled | an odd number is rolled)

iii) P(a prime number is rolled | an even number is rolled)

iv) P(an even number is rolled | a prime number is rolled)

v) P(an odd number is rolled | an even number is rolled)

12. A month is selected at random from the months of the year.

Let J = {month begins with the letter J} and Y = {month ends with the letter Y}.

Determine the following.

a) $P(J)$ **b)** $P(Y)$ **c)** $P(J|Y)$ **d)** $P(Y|J)$

e) $P(Y'|J)$ **f)** $P(Y|J')$ **g)** $P(Y'|J')$ **h)** $P(J'|Y')$

Multiple Choice **13.** Consider two events such that $P(A) = 0.3$, $P(B) = 0.4$, and $P(A \text{ or } B) = 0.58$. The events A, B are

A. dependent and mutually exclusive

B. dependent and not mutually exclusive

C. independent and mutually exclusive

D. independent and not mutually exclusive

Numerical Response **14.** A triangular spinner has 3 equal sections coloured red, blue, and green. To the nearest hundredth, the probability of the spinner not landing on red in both of two consecutive spins is _____ .

(Record your answer in the numerical response box from left to right.)

Answer Key

1. a) dependent **b)** independent **c)** dependent **d)** independent

2. a) no, $P(A \cap B) \neq 0$ **b)** yes, $P(A \cap B) = P(A) \bullet P(B)$

3. a) no, $P(A \cup B) \neq P(A) + P(B)$ **b)** no, $P(A \cap B) \neq P(A) \bullet P(B)$

4. a) 0.75 **b)** 0.5 **c)** 0.7 **5. a)** $\frac{1}{4}$ **b)** $\frac{2}{5}$ **c)** $\frac{2}{3}$

6. $\frac{1}{18}$ **7. a)** 0.6375 **b)** 0.2125 **c)** 0.1125 **d)** 0.0375

8. a) $\frac{1}{16}$ **b)** $\frac{1}{169}$ **c)** $\frac{1}{4}$ **d)** $\frac{1}{16}$ **e)** $\frac{1}{2}$

9. a) $\frac{1}{17}$ **b)** $\frac{1}{221}$ **c)** $\frac{25}{102}$ **d)** $\frac{13}{204}$ **e)** $\frac{26}{51}$

10.a) 0.35 **b)** no, since $P(B \cap D) \neq P(B) \bullet P(D)$ $\{0.35 \neq 0.33)$

11.a) i) 1, 3, 5 **ii)** 2, 4, 6 **iii)** 2, 3, 5 **b) i)** $\frac{2}{3}$ **ii)** $\frac{2}{3}$ **iii)** $\frac{1}{3}$ **iv)** $\frac{1}{3}$ **v)** 0

12.a) $\frac{1}{4}$ **b)** $\frac{1}{3}$ **c)** $\frac{1}{2}$ **d)** $\frac{2}{3}$ **e)** $\frac{1}{3}$ **f)** $\frac{2}{9}$ **g)** $\frac{7}{9}$ **h)** $\frac{7}{8}$

13. D **14.**

0	.	4	4