Lesson 5: Combinations - Part One

Permutations and Combinations Lesson #5:
Combinations - Part One

Combinations

Part 1: As part of the Grade 12 English course, students are required to read the following three books in a three month period:

"The Grapes of Wrath", "The Wars", "The Bean Trees".
G W B

a) Due to previous late returns, Steve is only allowed to sign out one English book from the school library per month.
List all the different orders in which Steve could sign out the three books.

GWB WGB BWG
GBW WBG BGW 6 ways

b) Tariq is allowed to sign out all three books at the same time.
How many different ways can he sign out all three books at the same time?

only 1 way

Part a) is an example of a permutation where the order is important.
Part b) is an example of a combination where the order is NOT important.

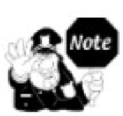
A selection of a set of elements in which the **order of the selection is NOT important** is called a **combination**.

Part 2: Suppose that the students in Part 1 were required to read only two of the books.

a) Complete the table to show the number of ways in which Steve and Tariq could do this:

Steve (Permutations)	Tariq (Combinations)
The Grapes of Wrath, The Wars The Grapes of Wrath, The Bean Trees The Wars, The Grapes of Wrath W, B B, G B W	The Grapes of Wrath, The Wars G,B W, B
6	3

b) Complete the following statement:
- The number of combinations is equal to the number of permutations divided by 2 or 2 factorial.
  2!



- A **permutation** is an **arrangement** of elements in which the order of the arrangement is taken into account.
- A **combination** is a **selection** of elements in which the order of selection is not taken into account.

Part 3: Five students, Al, Byron, Colin, Dave and Eric take part in a cross country race to represent their school.

a) Suppose the winner of the race wins $50, the runner-up wins $25, and third place runner wins $10.

The table below shows all possible ways in which the three prizes could be awarded to the five participants in the race.

"A" stands for Al, "B" for Byron, "C" for Colin, "D" for Dave, "E" for Eric.

ABC	ABD	ABE	ACD	ACE	ADE	BCD	BCE	BDE	CDE
ACB	ADB	AEB	ADC	AEC	AED	BDC	BEC	BED	CED
BAC	BAD	BAE	CAD	CAE	DAE	CBD	CBE	DBE	DCE
BCA	BDA	BEA	CDA	CEA	DEA	CDB	CEB	DEB	DEC
CAB	DAB	EAB	DAC	EAC	EAD	DBC	EBC	EBD	ECD
CBA	DBA	EBA	DCA	ECA	EDA	DCB	ECB	EDB	EDC

• Is this an example of permutations or combinations? permutation, order matters

• How many ways are there to award the three prizes?

$_5P_3 = 60$ ways

b) For participating in the cross-country race, the school has been awarded three places at a running clinic. The school coach decides to select the 3 lucky students from the ones who took part in the cross country race.

• Use the table from a) (which has been duplicated below) to circle the different ways the three students can be chosen.

ABC	ABD	ABE	ACD	ACE	ADE	BCD	BCE	BDE	CDE
~~ACB~~	~~ADB~~	~~AEB~~	~~ADC~~	~~AEC~~	~~AED~~	~~BDC~~	~~BEC~~	~~BED~~	~~CED~~
~~BAC~~	~~BAD~~	~~BAE~~	~~CAD~~	~~CAE~~	~~DAE~~	~~CBD~~	~~CBE~~	~~DBE~~	~~DCE~~
~~BCA~~	~~BDA~~	~~BEA~~	~~CDA~~	~~CEA~~	~~DEA~~	~~CDB~~	~~CEB~~	~~DEB~~	~~DEC~~
~~CAB~~	~~DAB~~	~~EAB~~	~~DAC~~	~~EAC~~	~~EAD~~	~~DBC~~	~~EBC~~	~~EBD~~	~~ECD~~
~~CBA~~	~~DBA~~	~~EBA~~	~~DCA~~	~~ECA~~	EDA	~~DCB~~	~~ECB~~	~~EDB~~	~~EDC~~

• Is this an example of permutations or combinations? combination
order does not matter

• How many ways are there to select the three students?

10

c) Complete the following statement:

• The number of combinations is equal to the number of permutations divided by 6 or 3! factorial.

Combinations of "n" different elements taken "r" at a time (r ≤ n)

The examples on the previous pages reflect the following general rule:

$$_nC_r = \frac{_nP_r}{r!} = \frac{n!}{(n-r)!\,r!}$$

The number of combinations of "n" elements taken "r" at a time is

$$_nC_r = \frac{n!}{(n-r)!\,r!}$$

$_5C_3 = \frac{5!}{(5-3)!\,3!} = \frac{5!}{2!\,3!}$

"n choose r"

- The $_nC_r$ key on a calculator can be used to evaluate combinations.
- In some texts $_nC_r$ is written as $\binom{n}{r}$. "n choose r"

Class Ex. #1

Three students from a class of ten are to be chosen to go on a school trip.

a) In how many ways can they be selected?
Write the answer in factorial notation and evaluate.

$_{10}C_3$ "10 choose 3" $= \frac{10!}{7!\,3!}$ $r = 10-3 = 7$

b) Confirm the answer in a) using the $_nC_r$ key on a calculator. $_{10}C_3 = 120$

Class Ex. #2

To win the LOTTO 649 a person must correctly select six numbers between 1 to 49.
Jasper selected the six numbers from the birth dates of his family, 3 7 9 11 20 29.
How many different selections of numbers could he have made?

$_{49}C_6 = 13{,}983{,}816$

Class Ex. #3

The Athletic Council decides to form a sub-committee of seven council members to look at how funds raised should be spent on sports activities in the school. There are a total of 15 athletic council members, 9 males and 6 females. The sub-committee must consist of exactly 3 females.

a) Determine the number of ways of selecting

i) the females $_6C_3 = 20$ **ii)** the males $_9C_4 = 126$ **iii)** the sub-committee $_6C_3 \cdot {_9C_4} = 2520$

b) In how many ways can the sub-committee be selected if Bruce, the football coach, must be included? (male)

$_6C_3 \cdot {_8C_3} \cdot {_1C_1} = 20 \cdot 56 \cdot 1 = 1120$

A standard deck of 52 cards has the following characteristics:

- 4 suits (spades, clubs, diamonds, and hearts).
- Each suit has 13 cards.
- Two suits are black (spades and clubs).
- Two suits are red (diamonds and hearts).
- Face cards are considered to be Jacks, Queens, and Kings.

Poker is a card game played from a deck of 52 cards.

a) How many different five card poker hands are possible?

$_{52}C_5 = 2,598,960$

b) In how many of the hands in a) will there be:

i) all diamonds?

$_{13}C_5 = 1287$

ii) 4 black cards and 1 red card?

$(_{26}C_4)(_{26}C_1)$
$=(14950)(26) = 388\,700$

iii) 3 kings and 2 queens?

$(_4C_3)(_4C_2) = (4)(6)$
$= 24$

iv) 3 kings? and 2 not king

$(_4C_3)(_{48}C_2)$
$(4)(1128)$
$= 4512$

v) four aces? and 1 non-ace

$(_4C_4)(_{48}C_1)$
$=(1)(48)$
$= 48$

vi) 5 cards of the same suit? (called a "flush")

5 clubs or 5 D, or 5 H or 5 Spades

$(_{13}C_5) \cdot 4 = 5148$

Complete Assignment Questions #1 - #12

#1, 2, 4, 5, 7

Assignment

1. Pete's Perfect Pizza Company has 9 choices of topping available.

a) How many different 2-topping pizzas can be made?

b) How many different 3-topping pizzas can be made?

2. A theatre company consisting of 6 players is to be chosen from 15 actors. How many selections are possible if the company must include Mrs. Jones?

3. How many different rectangles can be formed by eight horizontal lines and three vertical lines?

4. Edinburgh High School has a twelve-member student council. A four member sub-committee is to be selected to organize dances.

a) How many different sub-committees are possible?

b) How many four member sub-committees are possible if the council president and vice-president must be members?

5. A basketball coach has five guards and seven forwards on his basketball team.

a) In how many different ways can he select a starting team of two guards and three forwards?

b) How many different starting teams are there if the star player, who plays guard, must be included?

6. Twelve face cards are removed from a deck of fifty-two cards. From the face cards, three card hands are dealt. Determine the number of distinct three card hands that are possible which include

a) no restrictions

b) 3 kings

c) 1 Queen and 2 kings

d) exactly 1 Jack

7. Consider a standard deck of 52 cards. Determine the number of distinct six card hands that are possible which include

a) no restrictions

b) only clubs

c) 2 clubs and 4 diamonds

d) no sevens

e) 4 tens

f) exactly 1 Jack and 4 Queens

8. Explain the meaning of $\binom{10}{2}$. Why does $\binom{2}{10}$ not make sense?

9. Develop a problem where ${}_9C_4$ would be applicable as a solution.

Multiple Choice

10. There are 16 students in a class. The number of ways in which four students can be chosen to complete a survey is

A. $4!$ **B.** $\frac{16!}{4!}$

C. $\frac{16!}{12!\,4!}$ **D.** $\frac{16!}{12!}$

11. There are three girls and six boys on Leven High School softball team. Each of the students is capable of playing any fielding position on the team. There are nine fielding positions: a pitcher, a catcher, four infielders (first base, second base, third base, shortstop), and three outfielders (left field, centre field, right field).

For a particular game, Leven High School is in the field first. If the pitcher must be a girl and the catcher must be a boy, how many different positional line-ups are possible at the start of the game?

A. ${}_3C_1 \times {}_6C_1 \times 7!$
B. ${}_3C_1 \times {}_6C_1 \times 9!$
C. ${}_3C_1 \times {}_6C_1 \times {}_7C_4 \times {}_3C_3$
D. ${}_3C_1 \times {}_6C_1$

Numerical Response

12. Sarah is one of a group of eight people from which a committee of four people must be formed. The number of different committees possible if Sarah must sit on the committee is ______ .

(Record your answer in the numerical response box from left to right.)

Answer Key

1. a) 36 b) 84 **2.** 2002 **3.** 84 **4.** a) 495 b) 45
5. a) 350 b) 140 **6.** a) 220 b) 4 c) 24 d) 112
7. a) 20 358 520 b) 1716 c) 55 770 d) 12 271 512 e) 1128 f) 176
8. The number of ways of selecting 2 items from 10 where the order of selection is not important. You cannot select 10 items from 2.
9. Answers may vary. **10.** C **11.** A **12.**

3	5		

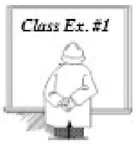
Lesson 6: Combinations - Part Two

$_{12}C_3$ = 12 choose 3, order does not matter

Permutations and Combinations Lesson #6:

Combinations - Part Two

Combinations Problems with "at least", "at most", etc.



The Student Council decides to form a sub-committee of five council members to look at how funds raised should be spent on the students of the school. There are a total of 11 student council members, 5 males and 6 females.
How many different ways can the five member sub-committee be chosen if there are

a) no restrictions on the number of males or females?

$_{11}C_5 = 462$

b) exactly three females?

3F 2M

$_6C_3 \cdot {_5C_2}$

$= 20 \cdot 10 = 200$

c) at least three females?

3F2M or 4F1M or 5F0M

$_6C_3 \cdot {_5C_2} + {_6C_4} \cdot {_5C_1} + {_6C_5} \cdot {_5C_0}$

$= 20 \cdot 10 + 15 \cdot 5 + 6 \cdot 1$

$= 200 + 75 + 6 = 281$

Consider a standard deck of 52 cards. How many different five card hands can be formed with

a) no conditions

$_{52}C_5$

$= 2{,}598{,}960$

b) at least 4 red cards?

4R1B or 5R0B

$_{26}C_4 \cdot {_{26}C_1} + {_{26}C_5}$

$= 14950 + 65780$

$= 80730$

c) at most 2 kings?

$\bar{K}$ = not king

0K5$\bar{K}$ or 1K4$\bar{K}$ or 2K3$\bar{K}$

• 4 Kings, 48 non Kings

$_{48}C_5 + {_4C_1} \cdot {_{48}C_4} + {_4C_2} \cdot {_{48}C_3}$

$= 1712304 + 4 \cdot 194580 + 6 \cdot 17296$

$= 2{,}594{,}400$

not

Using the Complement to Solve "at least" and "at most" Problems

Sometimes using complementary outcomes can reduce the workload in "at least" or "at most" problems.

Consider Class Ex. #1, in which there are 462 ways in which a five member sub-committee can be chosen from 5 males and 6 females. Students were asked to determine how many of the 462 possible sub-committees contained at least one female.

a) Brandon used the methods of Class Ex. #1 to solve the problem. Complete his work which has been started below.

\# of ways to select at least one female = # of ways to select 1 F and 5 M + # of ways to select 2 F and 3 M + # of ways to select 3 F and 2 M + # of ways to select 4 F and 1 M + # of ways to select 5 F and 0 M

$= {}_6C_1 \times {}_5C_4 + {}_6C_2 \times {}_5C_3 + {}_6C_3 \times {}_5C_2 + {}_6C_4 \times {}_5C_1 + {}_6C_5 \times {}_5C_0$

b) Alysha realized that there was a quicker way to solve the problem by using complementary outcomes. She partitioned the 462 sub committees into 6 groups. Complete her work which has been started below.

Total # of ways to select all subcommittees = # of ways to select 0 F and 5 M + # of ways to select 1 F and 4 M + # of ways to select 2 F and 3 M + # of ways to select 3 F and 2 M + # of ways to select 4 F and 1 M + # of ways to select 5 F and 0 M

Total # of ways to select all subcommittees = # of ways to select 0 F and 5 M + (# of ways to select 1 F and 4 M + # of ways to select 2 F and 3 M + # of ways to select 3 F and 2 M + # of ways to select 4 F and 1 M + # of ways to select 5 F and 0 M)

Total # of ways to select all subcommittees = # of ways to select 0 F and 5 M + (# of ways to select at least one female)

\# of ways to select at least one female = Total # of ways to select all subcommittees − # of ways to select 0 F and 5 M

\# of ways to select at least one female = 462 − ____

\# of ways to select at least one female = ____

- In Alysha's work, the outcome (0 females and 5 males) is the **complement** of the required outcome (at least one female).
- In general, A U − Ā = A

# of ways for the required outcome
= # of ways with no restrictions − # of ways for the complement of the required outcome

- When writing the solution to a similar problem, the first three lines of Alysha's work can be omitted.

Consider a standard deck of 52 cards. How many different five card hands can be formed containing at least 1 club?

• 1C4C̄ or 2C3C̄ or 3C2C̄ or 4C1C̄ or 5C

• all the possible hands − 0C5C̄

$_{52}C_5 - {}_{13}C_0 \cdot {}_{39}C_5 = 2{,}023{,}203$

no clubs

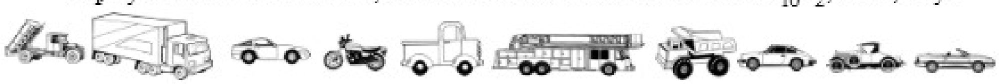
Complete Assignment Questions #1 - #5

#1-5

Combinations Which are Equivalent

Fran and Bernie have identical collections of vintage toy vehicles.

- Fran selects 2 toys from the 10 shown for display in her cabinet. Since the order of display does not matter to her, she calculates that this can be done in ${}_{10}C_2$, or 45, ways.



- Bernie selects 8 toys from the 10 shown for display in his cabinet. Since the order of display does not matter to him, he calculates that this can be done in ${}_{10}C_8$, or 45, ways.

a) Explain in words why ${}_{10}C_2 = {}_{10}C_8$.

b) Use factorial notation to show that ${}_{10}C_2 = {}_{10}C_8$.

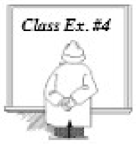
c) Give another two examples of equivalent combinations.

The above examples are representations of the general rule

$${}_nC_r = {}_nC_{n-r} \quad \text{or} \quad \binom{n}{r} = \binom{n}{n-r}$$

d) Algebraically show that ${}_nC_r = {}_nC_{n-r}$.

Solving for "n" in Combination Problems



During a Pee Wee hockey tryout, all the players met on the ice after the last practice and shook hands with each other. There were a total of 300 handshakes.

a) Write an equation involving combinations whose solution would determine the number of players on the ice.

b) Solve the equation in a)

i) by guess and check

ii) by using factorials

Complete Assignment Questions #6 - #9

Assignment

1. The Athletic Council decides to form a sub-committee of 6 council members to look at a new sports program. There are a total of 15 Athletic Council members, 6 females and 9 males. How many different ways can the sub-committee consist of at most one male?

2. A group of 4 journalists is to be chosen to cover a murder trial. There are 5 male and 7 female journalists available. How many possible groups can be formed

a) consisting of 2 men and 2 women? **b**) consisting of at least 3 men?

c) consisting of at least 1 woman?

3. Consider a standard deck of 52 cards. How many different four card hands have

a) at least one black card?

b) at least two kings?

c) at most two clubs?

4. City Council decides to form a sub-committee of five aldermen to investigate transportation concerns. There are 4 males and 7 females. How many different ways can the sub-committee be formed consisting of at least one female member?

5. An all-night showing at a movie theatre is to consist of five movies. There are fourteen different movies available, ten disaster movies and four horror movies. How many possible schedules include:

a) at least one horror movie?

b) at least four disaster movies?

c) both "Airport Disaster" and "Halloween Horror"?

6. Use "guess and check" on a calculator to determine the solution(s) to the following equations.

a) $\binom{n}{2} = 105$ **b)** ${}_nC_3 = 84$ **c)** ${}_{11}C_n = 330$

Multiple Choice **7.** After everyone had shaken hands once with everyone else in a room, there was a total of 66 handshakes. How many people were in the room?

A. 11
B. 12
C. 33
D. 67

8. There are 91 different ways of selecting two students from a class of students. Which of the following equations can be used to determine the number, n, of students in the class?

A. $n^2 - n - 182 = 0$
B. $n^2 - n - 91 = 0$
C. $n^2 - n + 182 = 0$
D. $n^2 - n + 91 = 0$

Numerical Response **9.** The number of ways that a selection of 7 students can be chosen from a class of 28 is the same as the number of ways that n students can be chosen from the same class. The value of n is _____ .

(Record your answer in the numerical response box from left to right.)

Answer Key

1. 55

2. a) 210 **b)** 75 **c)** 490

3. a) 255 775 **b)** 6961 **c)** 258 856

4. 462 **5. a)** 1750 **b)** 1092 **c)** 220

6. a) 15 **b)** 9 **c)** 4 or 7

7. B **8.** A **9.**

2	1		